Lecture Notes On

CENG 277

Discrete Structures

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Based on the book: Discrete and Combinatorial Mathematics, Ralph P. Grimaldi, 5th ed.

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Week 1– Counting - I

Union Rule: $|S \cup T| = |S| + |T| - |S \cap T|$.

Exercise 1-1: How many integers in $A = \{1, 2, \ldots, 300\}$ are divisible by 3 but not divisible by 7?

Product Rule: For finite sets $S_1, S_2, \ldots, S_k$ we have

$|S_1 \times S_2 \times \cdots \times S_k| = |S_1| \cdot |S_2| \cdots |S_k|$.

Exercise 1-2: Let $\Sigma = \{a, b, c, d, e, f, g\}$ How many words of 5 letters can we make

a) With repetition?

b) Without repetitions?

Exercise 1-3: Let $|S| = n, \quad |T| = m$. How many functions are there from S to T? How many are $1-1$?

Permutation: A permutation of a finite set is an ordered list of its elements, with each element occurring once.

$$P(n, r) = \frac{n!}{(n-r)!}$$

Combination: A combination of n objects taken r at a time is choosing an r element subset.

$$C(n, r) = \binom{n}{r} = \frac{n!}{(n-r)!r!}$$

Exercise 1-4: There are 15 girls and 18 boys in a class of 33. In how many different ways can we choose:

a) 5 students?

b) 2 boys and 3 girls?

Exercise 1-5: In how many ways can we order a group of 7 people such that

a) A and B are next to each other?

b) A and B are not next to each other?

c) A and B are next to each other but A and C are not?

Exercise 1-6: For 5 digit integers from 10000 to 99999, how many are palindromes and what is their sum?

$\Big($**Answer:** $900, 49\,500\,000\Big)$

Exercise 1-7: How many 11-letter words can be made from the letters of the word ABRACADABRA?

$\left(\textbf{Answer: } 83160\right)$

Exercise 1-8: How many $n-$bit sequences contain exactly k zeros?

Exercise 1-9: How many paths are there from one end of the chessboard to the other, if we can only move right and up?

Binomial Theorem: For $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$, we have:

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$$

We can prove this using induction and $\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$, but we can also prove it by counting.

Exercise 1-10: Prove that $\sum_{i=0}^{n} \binom{n}{i} = 2^n$. (Three different ways)

Exercise 1-11: We will select 4 distinct numbers from the list $\{-5, -4, -3, -2, -1, 1, 2, 3, 4\}$. In how many ways can we do this if their product is positive?

$\left(\textbf{Answer: } \binom{4}{4} + \binom{4}{2}\binom{5}{2} + \binom{5}{4} = 66\right)$

Exercise 1-12: In how many different ways can we express 72 as a sum of 2's and 3's if

a) Order is not important?

b) Order is important?

Solution:

a) Possible values for number of 2's: $\{0, 3, 6, 9, \ldots, 36\}$. Possible values for number of 3's: $\{24, 22, 20, \ldots, 0\}$. So there are clearly 13 possibilities.

b) If we use n 2's and m 3's, and order is important, it means we are permuting $n+m$ objects. But n of these and m of these can be permuted among themselves, so there are

$$\frac{(n+m)!}{n!\,m!}$$

different permutations (for this case). Therefore the answer is:

$$1 + \frac{35!}{2!\,33!} + \frac{34!}{4!\,30!} + \frac{33!}{6!\,27!} + \cdots + \frac{25!}{22!\,3!} + 1$$

$$= \sum_{k=0}^{12} \frac{(36-k)!}{(2k)!\,(36-3k)!}$$

Week 2– Counting - II

Objects in Boxes: If we want to place n identical objects into k distinct boxes, there are

$$\binom{n+k-1}{n}$$

different ways. We can prove this by considering permutations of n objects and $k-1$ dividers.

Exercise 2-1: How many numbers between 1 and 1000 have the property that their digits sum to 5?

$\left(\textbf{Answer:} \text{ Place 5 marbles into 3 boxes, } \binom{7}{5}\right)$

Exercise 2-2: We will select 8 pizzas from 4 available types. In how many ways can we make this choice?

Exercise 2-3: Twelve \$100 bills will be distributed to 4 people.

a) In how many ways can we do this?

b) How does the answer change if each person must receive at least \$200?

$\left(\textbf{Answer: } 455, 35\right)$

Exercise 2-4: **a)** How many ways are there to put 14 objects in 3 boxes with at least 8 objects in one box?

b) How many ways are there to put 14 objects in 3 boxes with no more than 7 objects in one box?

c) How many numbers between 0 and 999 have the sum of their digits equal to 20?

$\left(\textbf{Answer: } 84, 36, 36\right)$

Exercise 2-5: You can buy a pizza with 5 toppings. There are 8 possibilities for toppings: Pepperoni, ham, chicken, tuna, mushroom, shrimp, onion, olives.

a) How many different orders are possible if you buy a single pizza?

b) How many different orders are possible if you buy 3 pizzas?

Solution: **a)** $\binom{8}{5} = 56$

b) Distribute 3 orders to 56 pizza types: $\binom{56+3-1}{3} = \binom{58}{3} = 30856$

Note: The answer $\dfrac{56^3}{3!} = 29269.33$ is close, but obviously wrong. It is not even an integer! The error is that, if two among the three pizzas are the same, we are not counting this choice 6 times. We count it only 3 times.

Exercise 2-6: Find the number of solutions to $x_1 + x_2 + \cdots + x_{13} \leqslant 10$ where each x_i is integer and $x_i \geqslant 0$

$\left(\textbf{Answer: } \binom{10+14-1}{10} = \frac{23!}{10!\,13!}\right)$

Exercise 2-7: Find the number of solutions to $x_1 + x_2 + \cdots + x_{13} = 10$ where each x_i is integer and $x_i \geqslant 0$

$\left(\textbf{Answer: } \binom{10+13-1}{10} = \frac{22!}{10!\,12!}\right)$

Exercise 2-8: How many numbers in the set $\{10000, 10001, 10002, \ldots, 99999\}$ have sum of digits equal to 20?

Solution: Distribute 20 balls to 5 containers randomly. Then subtract the cases where one gets 10 or more. Then add cases where two gets 10.

$$\binom{24}{20} - \binom{5}{1}\binom{14}{10} + \binom{5}{2}$$

But this number includes the cases where the first digit is zero. So we have to find all such distributions for 4 containers and subtract:

$$\left[\binom{24}{20} - \binom{5}{1}\binom{14}{10} + \binom{5}{2}\right] - \left[\binom{23}{20} - \binom{4}{1}\binom{13}{10} + \binom{4}{2}\right] = 4998$$

An easier alternative would be to put one ball to the first container, then distribute remaining 19 randomly. This method results in:

$$\binom{23}{19} - \binom{4}{1}\binom{13}{9} - \binom{14}{10} + \binom{4}{1} = 4998$$

Exercise 2-9: Find the number of different integers n such that $1000 \leqslant n \leqslant 9999$ and the sum of digits of n is 9.
(For example 3033, 6012, 1223 etc.)

$\left(\textbf{Answer: } \binom{11}{8} = \binom{12}{9} - \binom{11}{9} = 165\right)$

Exercise 2-10: There are 7 departments in a faculty. We will choose 40 students from the faculty, such that there will be at least 3 and at most 20 students from any department. In how many different ways can we do this?

Solution: Give 3 to each. Distribute remaining 19 to 7 randomly, using 19 balls and 6 lines:

$$\frac{25!}{19!\,6!} = 177\,100$$

Now subtract those cases where one gets 19 (total 22) or 18 (total 21):

$$177\,100 - 7 - 7 \cdot 6 = 177\,051$$

Exercise 2-11: Consider all 4 digit numbers made from digits $\{1, 2, 3, 4, 5, 6, 7\}$. How many have nondecreasing digits?
(1234, 1222, 4444, 5667 are possible, 1231, 7512, 3776, 5553 are NOT)

Solution: We can choose and order:
4 distinct numbers in $\binom{7}{4} = 35$ different ways. (Example: 1246)

3 distinct numbers in $3\binom{7}{3} = 105$ different ways. (Example: 1244,1224,1124)

2 distinct numbers in $3\binom{7}{2} = 63$ different ways. (Example: 1112, 1122,1222)

1 number in 7 different ways. (Example: 1111)

So the answer is: $35 + 105 + 63 + 7 = 210$

Second Method:

Suppose $\boxed{a \mid b \mid c \mid d}$ is a solution. Consider $\boxed{a \mid b-a \mid c-b \mid d-c}$. All these numbers are positive or zero, their sum is at most 7. So this is a problem of distributing 7 balls to 5 children (one takes extra balls) such that the first gets at least 1.

$$\frac{(6+5-1)!}{6!\,4!} = 210$$

Ordered Partitions: If a set has n elements and if $n_1 + n_2 + \cdots + n_k = n$, then there are

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

ordered partitions, where the i^{th} partition has n_i elements.

Exercise 2-12: There are 30 students in a classroom. The instructor will form 4 groups of sizes 4, 7, 9 and 10, and give them homeworks from logic, sets, divisibility and counting. In how many ways can this be done?

Multinomial Theorem: The expression $\dfrac{n!}{n_1!n_2!\cdots n_k!}$ is called a multinomial coefficient and written as $\begin{pmatrix} & n & \\ n_1\, n_2 & \cdots & n_k \end{pmatrix}$.

For $x_1, x_2, \ldots, x_k \in \mathbb{R}$ and $n \in \mathbb{N}$ we have

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{x_1+\cdots x_k=n} \begin{pmatrix} & n & \\ n_1\, n_2 & \cdots & n_k \end{pmatrix} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$$

Exercise 2-13: Find the coefficient of x^8 in the expansion of $(1 + x^2 + x^4)^{10}$.

$\left(\textbf{Answer: } \dfrac{10!}{4!\,6!} + \dfrac{10!}{2!\,8!} + \dfrac{10!}{2!\,1!\,7!} = 615 \right)$

Exercise 2-14: Find the coefficient of $x^2y^6z^8$ in the expansion of $(x - 5y + z^4)^{10}$.

$\left(\textbf{Answer: } \dfrac{10!\,(-5)^6}{2!\,6!\,2!} \right)$

Week 3– Logic

Proposition: A proposition is any sentence that is either true or false.

Negation: If P is a statement, the negation of P ($\neg P$) is the statement "P is false."

Let P and Q be propositions. $P \wedge Q$ means both P and Q are true, $P \vee Q$ means P is true or Q is true. (or both).

Truth Tables: We can summarize the possible values of a compound statement using a truth table. Here, 1 means "True" and 0 means "False".

P	$\neg P$
1	0
0	1

P	Q	$P \wedge Q$
1	1	1
1	0	0
0	1	0
0	0	0

P	Q	$P \vee Q$
1	1	1
1	0	1
0	1	1
0	0	0

Logically Equivalent Statements: If two statements are both true or both false, we say they are logically equivalent. For example, $\neg(P \wedge Q)$ is logically equivalent to $\neg P \vee \neg Q$ and $\neg(P \vee Q)$ is logically equivalent to $\neg P \wedge \neg Q$.

Exercise 3-1: Show that $p \wedge (q \vee r)$ is logically equivalent to $(p \wedge q) \vee (p \wedge r)$.

Tautologies and Contradictions: A statement that is always true is called a tautology, a statement that is always false is called a contradiction. For example, $P \vee \neg P$ is a tautology, $P \wedge \neg P$ is a contradiction.

Implication and Biconditional: The statement $P \rightarrow Q$ (P implies Q) is the statement "If P is correct, then Q is correct". The statement $P \iff Q$ (P if and only if Q) means P and Q are logically equivalent, that is, they are either both true or both false.

P	Q	$P \rightarrow Q$
1	1	1
1	0	0
0	1	1
0	0	1

P	Q	$P \iff Q$
1	1	1
1	0	0
0	1	0
0	0	1

Exercise 3-2: Construct a truth table for the following:

a) $(p \wedge q) \rightarrow p$

b) $(p \rightarrow q) \rightarrow (q \rightarrow p)$

Exercise 3-3: Show that $p \rightarrow q$ is logically equivalent to $\neg p \vee q$.

The Laws of Logic

- $\neg\neg p \iff p$
- $\neg(p \vee q) \iff \neg p \wedge \neg q$
  $\neg(p \wedge q) \iff \neg p \vee \neg q$
- $p \vee q \iff q \vee p, \quad p \wedge q \iff q \wedge p,$
- $p \vee (q \vee r) \iff (p \vee q) \vee r, \quad p \wedge (q \wedge r) \iff (p \wedge q) \wedge r$
- $p \vee (q \wedge r) \iff (p \vee q) \wedge (p \vee r), \quad p \wedge (q \vee r) \iff (p \wedge q) \vee (p \wedge r)$
- $p \vee p \iff p, \quad p \wedge p \iff p$
- $p \vee \neg p \iff$ Tautology, $p \wedge \neg p \iff$ Contradiction

Exercise 3-4: Simplify $(p \vee q) \wedge \neg(\neg p \wedge q)$

Converse: $Q \to P$ is the converse of $P \to Q$.

Contrapositive: $\neg Q \to \neg P$ is the contrapositive of $P \to Q$. If an implication is correct, can we say its converse and contrapositive are also correct?

Exercise 3-5: Show that $P \iff Q$ is logically equivalent to $(P \to Q) \wedge (Q \to P)$

Exercise 3-6: What is the negation of $P \to Q$?

Universal and Existential Quantifiers: $\forall x P(x)$ means the statement $P(x)$ is correct for all x. $\exists x P(x)$ means there exists at least one x for which the statement $P(x)$ is correct.

$$\neg(\forall x \; P(x)) \iff \exists x \; \neg P(x)$$

$$\neg(\exists x \; P(x)) \iff \forall x \; \neg P(x)$$

Exercise 3-7: Find the negation of the following:
a) All odd numbers are prime.
b) Some 4th year students are taking this course.

Exercise 3-8: Show that $P \vee (Q \wedge R)$ is equivalent to $(P \vee Q) \wedge (p \vee R)$

Exercise 3-9: Prove or disprove the following:
a) $\forall x \in \mathbb{R}, (x+1)^2 \geqslant x^2$
b) $\exists n \in \mathbb{Z}, 5|n^2 - 2$
c) $\forall n \in \mathbb{Z}, 6|n^4 - n^2$

Exercise 3-10: Let P be the statement *n is a multiple of 4* and Q be *n^2 is a multiple of 4*. Which of the following are true?

a) $P \to Q$

b) $Q \to P$

c) $P \iff Q$

d) $\neg P \to \neg Q$

e) $\neg Q \to \neg P$

(**Answer:** a), e) are true, the others are false).

Exercise 3-11: Give converse and contrapositive of following statements:

a) $p \Rightarrow (q \wedge r)$

b) If $x + y < 10$ then $x^2 + y^2 < 100$

Solution:

a) Converse: $(q \wedge r) \Rightarrow p$
Contrapositive: $\neg q \vee \neg r \Rightarrow \neg p$

b) Converse: If $x^2 + y^2 < 100$then $x + y < 10$
Contrapositive: If $x^2 + y^2 \geqslant 100$ then $x + y \geqslant 10$

Exercise 3-12: Convert into logical notation: If he listens Shakira, he feels good. If Fenerbahçe wins, he feels good. He feels good now, therefore either he is listening to Shakira, or Fenerbahçe won today.
Is this correct?

(**Answer:** $[(p \Rightarrow r) \wedge (q \Rightarrow r)] \Rightarrow [r \Rightarrow (p \vee q)]$, No)

Exercise 3-13: Show that $p \to (q \vee r)$ is logically equivalent to $(p \wedge \neg q) \to r$

a) Using a truth table,

b) Without using a truth table.

Solution:

p	q	r	$q \vee r$	$p \to (q \vee r)$	$p \wedge \neg q$	$(p \wedge \neg q) \to r$
1	1	1	1	1	0	1
1	1	0	1	1	0	1
1	0	1	1	1	1	1
1	0	0	0	0	1	0
0	1	1	1	1	0	1
0	1	0	1	1	0	1
0	0	1	1	1	0	1
0	0	0	0	1	0	1

Fifth and seventh columns are identical.
Alternatively, we can show equivalence as:

$$\begin{aligned}
p \to (q \vee r) &\iff \neg p \vee (q \vee r) \\
&\iff (\neg p \vee q) \vee r \\
&\iff \neg(p \wedge \neg q) \vee r \\
&\iff (p \wedge \neg q) \to r
\end{aligned}$$

Exercise 3-14: Show that $(p \vee q) \to (q \to q)$ is a tautology

a) Using a truth table,

b) Without using a truth table.

Solution:

p	q	$p \vee q$	$q \to q$	$(p \vee q) \to (q \to q)$
1	1	1	1	1
1	0	1	1	1
0	1	1	1	1
0	0	0	1	1

Alternatively, we can show tautology as:

$$\begin{aligned}
(p \vee q) \to (q \to q) &\iff (p \vee q) \to (\neg q \vee q) \\
&\iff (p \vee q) \to 1 \\
&\iff \neg(p \vee q) \vee 1 \\
&\iff 1
\end{aligned}$$

Exercise 3-15: For the following statements the universe is real numbers. Are they correct? Explain.

a) $\forall x \exists y (x^2 + y^2 = 1)$

b) $\exists x \forall y (x^2 + y^2 = 1)$

c) $\exists x \exists y (x^2 + y^2 = 1)$

Solution:

a) NO.
If $x = 2$ there is no $y \in \mathbb{R}$ such that $x^2 + y^2 = 1$.

b) NO.
There is no $x \in \mathbb{R}$ such that $x^2 + 5^2 = 1$.

c) YES
$x = 0, y = 1$ is one of the possible choices satisfying $x^2 + y^2 = 1$.

Exercise 3-16: Express the following in logical notation. Is the reasoning correct?

- If there is a homework on Friday, I do not go to the movies on weekend.
- If I go to the movies on weekend, I have no money left on Monday.
- I have no money left on Monday. So there was no homework on Friday.

Solution: f: There is a homework on Friday
w: I go to the movies on Weekend
m: I have some money left on Monday

The given statements are:

$$\begin{array}{rcl} f & \longrightarrow & \neg w \\ w & \longrightarrow & \neg m \\ \neg m & \longrightarrow & \neg f \end{array}$$

The reasoning is wrong. It is possible that f is TRUE, w is FALSE and m is FALSE. This makes the first two assumption TRUE but the conclusion FALSE.

Exercise 3-17: A student says *If I miss the bus and there is an exam in the morning, then I am in trouble.* Express this in logical notation. Find the contrapositive of this statement.

Solution:
P : I miss the bus
Q : There is an exam in the morning
R : I am in trouble
The statement is equivalent to:

$$(P \wedge Q) \rightarrow R$$

Its contrapositive is:

$$\neg R \rightarrow (\neg P \vee \neg Q)$$

In other words, *If I am not in trouble, then either I didn't miss the bus or I did not have an exam in the morning.*

Exercise 3-18: Let m, n be positive integers. Are the following statements true or false? (No explanation necessary)

a) $\forall n \; \exists m \; [n = m^2]$

b) $\forall n \; \exists m \; [m = n^2]$

c) $\exists n \; \exists m \; [n = m^2]$

d) $\exists n \; \forall m \; [m = n^2]$

Exercise 3-19: Show that $[(p \wedge \neg q) \wedge r] \rightarrow [(p \wedge r) \vee q]$ is a tautology or provide a counterexample.

Solution:

p	q	r	$p \wedge \neg q \wedge r$	$(p \wedge r) \vee q$	S
1	1	1	0	1	1
1	1	0	0	1	1
1	0	1	1	1	1
1	0	0	0	1	1
0	1	1	0	0	1
0	1	0	0	1	1
0	0	1	0	0	1
0	0	0	0	0	1

The statement is correct for all possible truth values of p, q, r. Therefore it is a tautology.

Exercise 3-20: Show that $\big[(p \wedge \neg q) \wedge [p \rightarrow (q \rightarrow r)]\big] \rightarrow \neg r$ is a tautology or provide a counterexample.

Solution: We can solve this question with a truth table. Or, alternatively, we can do this: If the implication is wrong, first part must be true, so $p = 1, q = 0$. The second part must be false, so $r = 1$. This is the counterexample we want: $(1, 0, 1)$.

The statement is NOT a tautology.

Week 4– Induction

Well-Ordering Principle: Every nonempty subset of $\mathbb{Z}^+$ contains a smallest element.

First Principle of Mathematical Induction: Let $P(n)$ be a proposition about the positive integer n. If

(I) $P(1)$ is true,

(II) $P(k+1)$ is true whenever $P(k)$ is true, (for $k \geqslant 1$)

then the propositions $P(n)$ are true for all positive integers.

Second Principle of Mathematical Induction: Let $P(n)$ be a proposition about the positive integer n. If

(I) $P(1)$ is true,

(II) $P(k+1)$ is true whenever $P(i)$ is true for every positive integer $i \leqslant k$

then the propositions $P(n)$ are true for all positive integers.
We can modify these principles such that the induction will start from n_0 rather than 1.

Exercise 4-1: Show that $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Exercise 4-2: Show that $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

Exercise 4-3: Show that $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9 for all $n \in \mathbb{Z}^+$.

Exercise 4-4: Show that $(x+y)^n = \displaystyle\sum_{i=1}^{n} \binom{n}{i} x^{n-i} y^i$

Exercise 4-5: Fibonacci sequence is defined as: $F_1 = 1, F_2 = 1, F_n = F_{n-1} + F_{n-2}$. The first few terms are $1, 1, 2, 3, 5, 8, 13, 21, \ldots$. Show that the formula

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right]$$

is correct for all n.

Exercise 4-6: Show that the sum of the interior angles of a polygon of n sides is $\pi(n-2)$. (Hint: Start with the triangle and use induction)

Exercise 4-7: Prove the following statements: $(n \in \mathbb{Z}^+)$

a) $16 | 5^n - 4n - 1$

b) $\displaystyle\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \geqslant \sqrt{n}$

c) $(2n)! \geqslant 2^n n!$

Exercise 4-8: Can we divide a class of n students into groups of 4 or 5?

Exercise 4-9: Prove that $5^{n+1} + 2 \cdot 3^n + 1$ is divisible by 8 for all $n \in \mathbb{N}$.

Exercise 4-10: Prove that $11^n - 4^n$ is divisible by 7 for all $n \in \mathbb{Z}^+$.

Exercise 4-11: Prove that $n! \geqslant 2^n$ for $n \geqslant 4$.

Exercise 4-12: Show that $8^{n+2} + 9^{2n+1}$ is divisible by 73 for $n \in \mathbb{N}$.

Solution: $n = 0 \quad \Rightarrow \quad 73 \mid 64 + 9$

Assume it is correct for $n = k$, in other words $73 \mid 8^{k+2} + 9^{2k+1}$. Then

$$8^{k+2} + 9^{2k+1} = 73p, \quad \text{where } p \text{ is an integer}$$

Now check the case $n = k + 1$:

$$\begin{aligned} 8^{k+3} + 9^{2k+3} &= 8 \cdot 8^{k+2} + 81 \cdot 9^{2k+1} \\ &= 8 \cdot 8^{k+2} + 8 \cdot 9^{2k+1} + 73 \cdot 9^{2k+1} \\ &= 8 \cdot 73p + 73 \cdot 9^{2k+1} \end{aligned}$$

Therefore $8^{k+3} + 9^{2k+3} = 73q$ for some integer q and the claim is correct by mathematical induction.

Exercise 4-13: Show that $4 + 10 + 16 + \cdots + (6n + 4) = (n + 1)(3n + 4)$ for all $n \in \mathbb{N}$.

Solution: $n = 0 \quad \Rightarrow \quad 4 = 4$
Assume it is correct for $n = k$:

$$4 + 10 + 16 + \cdots + (6k + 4) = (k + 1)(3k + 4)$$

Now check $n = k + 1$. Add $6(k + 1) + 4$ to both sides and rearrange to obtain

$$4 + 10 + 16 + \cdots + (6k + 4) + (6k + 10) = (k + 2)(3k + 7)$$

Therefore the formula is correct by mathematical induction.

Exercise 4-14: Show that $23 \,\big|\, 9 \cdot 2^{3n-1} + 10^{2n-1}$ for all positive integers n.

Solution: For $n = 1$, $9 \cdot 2^2 + 10^1 = 46$, $\quad 23 \,\big|\, 46$.
Assume the claim is correct for $n = k$. Then $9 \cdot 2^{3k-1} + 10^{2k-1} = 23p$ for some integer p.
For $n = k + 1$:

$$\begin{aligned} 9 \cdot 2^{3(k+1)-1} + 10^{2(k+1)-1} &= 9 \cdot 8 \cdot 2^{3k-1} + 100 \cdot 10^{2k-1} \\ &= 8 \cdot (9 \cdot 2^{3k-1} + 10^{2k-1}) + 92 \cdot 10^{2k-1} \\ &= 8 \cdot 23p + 4 \cdot 23 \cdot 10^{2k-1} \\ &= 23q \end{aligned}$$

for some integer q. Therefore by mathematical induction, $23 \,\big|\, 9 \cdot 2^{3k-1} + 10^{2k-1}$

Exercise 4-15: Show that $10 \mid n^5 - n$ for all $n \in \mathbb{Z}^+$.

Solution: Let $n = 1$. We know that $10 \mid 0$ so the statement is correct for $n = 1$.
Suppose it is correct for $n = k$, in other words, assume $10 \mid k^5 - k$, or $k^5 - k = 10p$ where p is an integer.

$$\begin{aligned} (k + 1)^5 - (k + 1) &= k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1 \\ &= (k^5 - k) + 5k(k^3 + 1) + 10(k^3 + k^2) \\ &= 10p + 5k(k^3 + 1) + 10(k^3 + k^2) \\ &= 10q + 5k(k^3 + 1) \end{aligned}$$

Where q is an integer. Either k is even, or $k^3 + 1$ is even, in both cases $5k(k^3 + 1)$ is a multiple of 10. Therefore

$$(k + 1)^5 - (k + 1) = 10r$$

For some integer r. In other words

$$10 \mid (k + 1)^5 - (k + 1) \quad \text{by mathematical induction.}$$

Exercise 4-16: Show that $(1+x)^n \geqslant 1+nx$ where $n = 1, 2, 3, \dots$ and $x \in \mathbb{R}^+$.

Solution: For $n = 1$ we have $(1+x) \geqslant 1+x$ so the statement is correct for $n = 1$.

Assume that it is correct for $n = k$, in other words:

$$(1+x)^k \geqslant 1+kx$$

Multiply both sides by $(1+x)$:

$$(1+x)^{k+1} \geqslant (1+kx)(1+x)$$

$$(1+x)^{k+1} \geqslant 1+kx+x+kx^2$$

If we eliminate kx^2 , the inequality will not change sides.

$$(1+x)^{k+1} \geqslant 1+(k+1)x$$

By mathematical induction, the formula is correct for all $n \in \mathbb{Z}^+$.

Exercise 4-17: Show that

$$1 \cdot 2 + 2 \cdot 3 + \cdots + n \cdot (n+1) = \frac{1}{3}n(n+1)(n+2)$$

for all $n \in \mathbb{Z}^+$.

Solution: For $n = 1$ we have $1 \cdot 2 = 2 = \frac{1}{3} \cdot 1 \cdot 2 \cdot 3$ or $2 = 2$ so the statement is correct for $n = 1$.

Assume that it is correct for $n = k$, in other words:

$$1 \cdot 2 + 2 \cdot 3 + \cdots + k \cdot (k+1) = \frac{1}{3}k(k+1)(k+2)$$

Add $(k+1)(k+2)$ to both sides:

$$\begin{aligned} 1 \cdot 2 + 2 \cdot 3 + \cdots + k \cdot (k+1) + (k+1) \cdot (k+2) &= \frac{1}{3}k(k+1)(k+2) + (k+1) \cdot (k+2) \\ &= (k+1)(k+2)\left[\frac{1}{3}k+1\right] \\ &= \frac{1}{3}(k+1)(k+2)(k+3) \end{aligned}$$

By mathematical induction, the formula is correct for all $n \in \mathbb{Z}^+$.

Week 5– Prime Numbers, Integers

Definition: If $n \in \mathbb{Z}^+$, $n > 1$ and n has exactly two divisors, 1 and n, it is called a prime number.

Theorem: (Euclid) There are infinitely many primes.

Exercise 5-1: Write an algorithm that makes a list of prime numbers up to a given number.

The Division Algorithm: Let $n \in \mathbb{Z}$ and $m \in \mathbb{Z}^+$. There are unique integers q and r such that

$$n = mq + r \qquad 0 \leqslant r < m$$

Definition: Let a, b be integers, not both zero. A positive integer c is called a **greatest common divisor** of a and b if

1) $c \mid a$ and $c \mid b$

2) If d is another common divisor of a and b, then $d \mid c$.

Theorem: Let $a, b \in \mathbb{Z}^+$. Then, there exists a unique $c \in \mathbb{Z}^+$ that is the greatest common divisor of a and b. Furthermore, there exist $x, y \in \mathbb{Z}$ such that $c = ax + by$.

Proof: Let $S = \{as + bt \mid s, t \in \mathbb{Z}, as + bt > 0\}$. Let c be the least element of S. Let's divide a by c.

$$a = qc + r$$

Then $r = a - qc = (1 - qx)a + (-qy)b$. If $r > 0, r \in S$ But also $r < c$ which contradicts the choice of c. Therefore $r = 0$ in other words $c \mid a$. Similarly, we can prove $c \mid b$.

If $\gcd(a, b) = 1$ we say a and b are relatively prime.

Exercise 5-2: Prove that $\gcd(n, n + 2) = 1$ or 2.

Euclidean Algorithm

Euclidean algorithm is a very fast way of finding gcd.

Theorem: If m and n are integers with $n > 0$ then the common divisors of m and n are the same as common divisors of n and $m \mod n$. Therefore $\gcd(m, n) = \gcd(n, r)$ where $r = m \pmod n$

Proof: Let $m = nq + r$. If $d \mid m,\ d \mid n$, then $d \mid r$ and conversely if $d \mid n,\ d \mid r$, then $d \mid m$. This means that, common divisors of (m, n) and common divisors of (n, r) are identical sets.

Algorithm:

Input: $m, n \in \mathbb{N}$, not both zero
Output: $\gcd(m, n)$
$a = m, b = n$
while $b \neq 0$ do
$(a, b) = (b, a \mod b)$
return a

Example: $m = 49$, $n = 19$

$$\begin{aligned} 49 - 2 \cdot 19 &= 11 \\ 19 - 1 \cdot 11 &= 8 \\ 11 - 1 \cdot 8 &= 3 \\ 8 - 2 \cdot 3 &= 2 \\ 3 - 1 \cdot 2 &= 1 \end{aligned}$$

We can see that $\gcd(49, 19) = 1$ which means they are relatively prime. Now reversing this process we obtain:

$$\begin{aligned} 1 &= 3 - 1 \cdot 2 \\ &= 3 - 1 \cdot (8 - 2 \cdot 3) \\ &= -1 \cdot 8 + 3 \cdot 3 \\ &= -1 \cdot 8 + 3 \cdot (11 - 1 \cdot 8) \\ &= 3 \cdot 11 - 4 \cdot 8 \\ &= 3 \cdot 11 - 4 \cdot (19 - 1 \cdot 11) \\ &= -4 \cdot 19 + 7 \cdot 11 \\ &= -4 \cdot 19 + 7 \cdot (49 - 2 \cdot 19) \\ &= 7 \cdot 49 - 18 \cdot 19 \end{aligned}$$

Thus we have found two numbers $(x, y) = (7, -18)$ such that $49x + 19y = 1$.

Exercise 5-3: Find $c = \gcd(111, 81)$. Then find $x, y \in \mathbb{Z}$ such that $111x + 81y = c$.

$\left(\textbf{Answer: } -8 \cdot 111 + 11 \cdot 81 = 3\right)$

Exercise 5-4: Find two integers m, n such that
$234m + 677n = 1$

$\left(\textbf{Answer: } m = 298,\ n = -103\right)$

Exercise 5-5: **a)** Find the greatest common divisor of 612 and 221.

b) Find two integers x, y such that $612x + 221y = c$, where $c = \gcd(612, 221)$.

Solution:

a)

$$\begin{aligned} 612 - 2 \cdot 221 &= 170 \\ 221 - 1 \cdot 170 &= 51 \\ 170 - 3 \cdot 51 &= 17 \\ 51 - 3 \cdot 17 &= 0 \end{aligned}$$

$$\gcd(612, 221) = 17$$

b)

$$\begin{aligned} 17 &= 170 - 3 \cdot 51 \\ &= 170 - 3 \cdot (221 - 170) \\ &= -3 \cdot 221 + 4 \cdot 170 \\ &= -3 \cdot 221 + 4 \cdot (612 - 2 \cdot 221) \end{aligned}$$

$$17 = 4 \cdot 612 - 11 \cdot 221$$

Theorem: Every integer $n > 1$ can be written as a product of primes uniquely, up to the order of primes.

Exercise 5-6: How many positive divisors does 240 have?

Exercise 5-7: How many positive divisors does 148500 have?

$\left(\textbf{Answer: } 96\right)$

Exercise 5-8: There are 100 prisoners in a prison, each in a separate cell. Each pressing of a button outside opens the door if it is locked and locks the door if it is open. On the day of a festival, the first guard presses each button to free the prisoners. Then, the second one presses $\{2, 4, 6, \ldots\}$, the third one $\{3, 6, 9, \ldots\}$ etc. until the 100th guard presses number 100. Which prisoners are free?

Exercise 5-9: Find smallest integer n such that $1260n$ is a cube.

Exercise 5-10: Find smallest integer m such that $2m$ is a square and $3m$ is a cube.

$\left(\textbf{Answer: } 72\right)$

Exercise 5-11: $(3,5,7)$ is a prime triple. Find another one if there is. Prove that there's no other if there is not.

$\left(\textbf{Answer: } \mod 3\right)$

Exercise 5-12: You have two containers, one is a 3 liter and the other is a 5 liter container. You have unlimited amount of water, and can transfer between the containers. Can you obtain amounts of 1,2,4 liters? What if we begin with 3 and 6?

Exercise 5-13: Find $c = \gcd(128, 38)$. Then find $x, y \in \mathbb{Z}$ such that $128x + 38y = c$.

$\left(\textbf{Answer: } -8 \cdot 128 + 27 \cdot 38 = 2\right)$

Exercise 5-14: Determine the smallest square number that is divisible by 8!.

Solution: $8! = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \quad \Rightarrow \quad n = 2 \cdot 5 \cdot 7 \cdot 8! = (2^4 \cdot 3 \cdot 5 \cdot 7)^2 = 1680^2$

Exercise 5-15: Find the number of divisors of 51975.

Solution: $51975 = 3^3 \cdot 5^2 \cdot 7 \cdot 11$

It has $4 \cdot 3 \cdot 2 \cdot 2 = 48$ divisors.

Exercise 5-16: Are the following statements true or false? Here, x, y, z are integers. Explain.

a) $\forall x\, \exists y\, \exists z\ (x = 3y + 8z)$

b) $\forall x\, \exists y\, \exists z\ (x = 2y + 8z)$

Solution: **a)** We can express the number 1 in terms of 3 and 8:

$$1 = 3 \cdot 3 - 1 \cdot 8$$

Therefore we can express any integer x in terms of 3 and 8.

$$x = 3 \cdot 3x - 1 \cdot 8x \quad \Rightarrow \quad \text{TRUE}$$

b) We can not express an odd integer in terms of 2 and 8. For example, if $x = 5$, there is no y, z, such that $5 = 2y + 8z \quad \Rightarrow \quad$ FALSE.

Week 6– Relations, Functions

Given sets S and T, a binary relation from S to T is a subset of $S \times T$.

$$R = \{(s,t) \mid s \in S, t \in T \text{ and } \ldots\}$$

If s is related to t we say sRt or $(s,t) \in R$.
Let R be a relation on S, that is , $R \subseteq S \times S$. We say R is:

Reflexive if $(x,x) \in R$ for all $x \in S$

Symmetric if $(x,y) \in R$ implies $(y,x) \in R$

Antisymmetric if $(x,y) \in R$ and $(y,x) \in R$ implies $x = y$

Transitive if $(x,y) \in R$ and $(y,z) \in R$ implies $(x,z) \in R$

Exercise 6-1: Let $S = \{1,2,3,4,5\}$. Find a relation R on S that is symmetric, transitive, not reflexive and not anti-symmetric.

$\Big($**Answer:** $\{(1,2),(2,1),(1,1),(2,2)\}\Big)$

Exercise 6-2: Let $S = \{1,2,3,4,5\}$. Find a relation R on S that is anti-symmetric, reflexive, not transitive and not symmetric.

$\Big($**Answer:** $\{(1,1),(2,2),(3,3),(4,4),(5,5),(1,2),(2,3)\}\Big)$

Equivalence Relations: A relation R on a set A is called an equivalence relation if it is reflexive, symmetric and transitive.

If R is an equivalence relation, we write $a \sim b$ (a is equivalent to b) instead of aRb.

Let $\sim$ be an equivalence relation on a set S. For each $s \in S$ we define the equivalence class of s as:

$$[s] = \{t \in S \mid t \sim s\}$$

Theorem: The following are logically equivalent:

I) $s \sim t$

II) $[s] = [t]$

III) $[s] \cap [t] \neq \varnothing$

Theorem: An equivalence relation determines a partition, and a partition determines an equivalence relation.

Exercise 6-3: For the following questions, determine whether the given relation is reflexive, symmetric, transitive, anti-symmetric:

a) The relation R is defined on the set of lines on the plane as $L_1 R L_2$ if $L_1 \perp L_2$.

$\Big($**Answer:** Symmetric$\Big)$

b) The relation R is defined on the set of lines on the plane as $L_1 R L_2$ if $L_1 \parallel L_2$.

$\Big($**Answer:** Reflexive, symmetric, transitive.$\Big)$

c) The relation R is defined on the set of people as $p_1 R p_2$ if p_1 and p_2 have a common parent.

$\Big($**Answer:** Reflexive and symmetric.$\Big)$

d) $R \subseteq \mathbb{Z} \times \mathbb{Z}, \quad mRn$ if $m = -n$.

$\Big($**Answer:** Symmetric.$\Big)$

e) $R \subseteq \mathbb{R} \times \mathbb{R}, \quad xRy$ if $|x - y| < 1$.

$\Big($**Answer:** Reflexive and symmetric.$\Big)$

f) R is a relation on $\mathcal{P}(S)$ for some set S, $(A, B) \in R$ if $A \subseteq B$.

$\Big($**Answer:** Reflexive, antisymmetric, transitive.$\Big)$

g) R is a relation on $\mathbb{N}$, $(m, n) \in R$ if $m \mid n$.

$\Big($**Answer:** Reflexive, antisymmetric, transitive.$\Big)$

Exercise 6-4: A relation R is defined on the Cartesian plane as $(x, y)R(z, w)$ if $xw = yz$. Is this an equivalence relation?

$\Big($**Answer:** Yes$\Big)$

Exercise 6-5: Let $n = abcd$ denote a four digit number.

a) Show that $9|n$ if and only if
$9|a + b + c + d$

b) Show that $11|n$ if and only if
$11|a - b + c - d$

$\Big($**Answer:** Consider n in mod 9 and mod 11.$\Big)$

FUNCTIONS

A function $f : A \longrightarrow B$ gives one and only one output from B for each input from A. The set A is called domain of f.

Composition: Function composition is defined as $(f \circ g)(x) = f(g(x))$.

In general $f \circ g \neq g \circ f$ but $(f \circ g) \circ h = f \circ (g \circ h)$. (Function composition is associative but it is not commutative)

One–to–one Functions: If $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ then f is one-to one.

Onto Functions: If there exists a $x \in A$ for all $y \in B$ such that $f(x) = y$ then f is onto.

Exercise 6-6: Find functions $f : \mathbb{N} \to \mathbb{N}$ that are:

a) One-to-one and onto.

b) One-to-one but not onto.

c) Onto but not one-to-one.

d) Not one-to-one and not onto.

Bijections: If a function is one–to–one and onto, we call it a one–to–one correspondence or a bijection.

Exercise 6-7: Let $f : B \to C$, $g : A \to B$, $f \circ g : A \to C$. Either show that the following propositions are true, or give counterexamples:

a) If $f \circ g$ is one-to-one, then, f is one-to-one.

b) If $f \circ g$ is one-to-one, then, g is one-to-one.

c) If $f \circ g$ is onto, then, f is onto.

d) If $f \circ g$ is onto, then, g is onto.

Theorem: Let $f : A \to B$ and $g : B \to C$.

a) If f and g are one-to-one, then $g \circ f$ is one-to-one.

b) f f and g are onto, then $g \circ f$ is onto.

Theorem: A function $f : A \to B$ is invertible if and only if it is one-to-one and onto.

Theorem: If $f : A \to B$ and $g : B \to C$ are invertible, then $g \circ f : A \to C$ is invertible and

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$$

Theorem: Let $f : A \to B$ where A, B are finite sets with $|A| = |B|$. Then, the following are equivalent:

a) f is one-to-one.

b) f is onto.

c) f is invertible.

Exercise 6-8: Let $f : [0,2] \to [-4,0],\ f(x) = x^2 - 4x$. Find the inverse of f.

$\left(\textbf{Answer: } f^{-1}(x) = 2 - \sqrt{4+x}\right)$

Exercise 6-9: For the following questions, describe a bijection (one–to–one correspondence) between the two given sets:

a) 40–bit sequences containing exactly 23 zeros and possible paths from $(0,0)$ to $(17,23)$ consisting of right and up steps of length 1.

b) n–bit sequences and all subsets of a set containing n elements.

c) 20–bit sequences containing exactly 4 one's and different ways of distributing 16 marbles to 5 different kids.

d) 110–bit sequences with exactly 10 ones and solutions over $\mathbb{N}$ to the equation $x_1 + x_2 + \cdots + x_{10} \leqslant 100$

Exercise 6-10: Let $A = \{1,2,3\}$ and $B = \{a,b\}$.

a) How many functions are there from A to B?

b) How many one-to-one functions are there from A to B?

c) How many onto functions are there from A to B?

d) How many one-to-one onto functions are there from A to B?

Exercise 6-11: Same question, this time for $|A| = m$ and $|B| = n$ (Number of elements).

Week 7– Pigeonhole Principle

If a finite set S is partitioned into k sets, then at least one of them has

$$\lceil |S|/k \rceil$$

or more elements.

Example: A sack contains 50 marbles of 4 different colors. There must be at least 13 marbles of the same color.

Exercise 7-1: How many times must we roll a single die to get the same score n times?

$\left(\textbf{Answer: } 6(n-1)+1 = 6n-5\right)$

Exercise 7-2: Let A be a 10–element subset of $\{1, 2, 3, \ldots, 50\}$. Show that A has two different 4–element subsets the sums of whose elements are equal.

$\left(\textbf{Answer: } \text{ Consider } \binom{10}{4}\right)$

Exercise 7-3: Show that in any set of 301 integers there must be a pair (m, n) such that $300|m-n$.

$\left(\textbf{Answer: } \text{ Use pigeonhole principle}\right)$

Exercise 7-4: We select 5 points from (the interior of) an equilateral triangle with side length 1. Show that there are at least two whose distance is less than 1/2.

Exercise 7-5: Show that if any 14 integers are selected from the set $S = \{1, 2, 3, \ldots 25\}$ there are at least two whose sum is 26.

Exercise 7-6: Let $A = \{1, 2, 3, \ldots, 10\}$ and $B = \{1, 2, 3, \ldots, 12\}$. How many one-to-one functions are there from A to B such that $f(1) = 1$?

$\left(\textbf{Answer: } 1 \cdot 11 \cdot 10 \cdot 9 \cdots 3 = \frac{11!}{2!}\right)$

Exercise 7-7: Let $A = \{1, 2, 3, \ldots, 10\}$ and $B = \{1, 2, 3, \ldots, 12\}$. How many functions are there from A to B such that $f(1)$ is even?

$\left(\textbf{Answer: } 6 \cdot 12^9\right)$

Exercise 7-8: The function $f : \mathbb{N} \to \mathbb{N}$ is given by $f(n) = 2^n$. Is this function one-to-one? Is it onto? Explain.

Solution: This function is one-to-one. Clearly, if $n_1 \neq n_2$ then $2^{n_1} \neq 2^{n_2}$
In other words $f(n_1) \neq f(n_2)$.

This function is NOT onto. There is no $n \in \mathbb{N}$ such that $2^n = 3$ or $2^n = 5, 6, 7, 9$ etc.

Exercise 7-9: Consider the set $S = \{1, 2, 3, \ldots, 20\}$ on the number line. Define a relation R on S as follows: two numbers x, y are related if $|x - y| \leqslant 2$. For example, $(2,3) \in R$, $\quad (2,4) \in R$ but $(2,5) \notin R$. Is this relation reflexive? Is it symmetric? Is it antisymmetric? Is it transitive?

Solution: The relation R is:

Reflexive, because $|x - x| = 0 \leqslant 2$

Symmetric, because if $|x - y| \leqslant 2$ then $|y - x| \leqslant 2$

NOT Antisymmetric, because $(2,3) \in R$ and $(3,2) \in R$.

NOT Transitive because $(2,4) \in R$ and $(4,6) \in R$, but $(2,6) \notin R$.

Exercise 7-10: There are 4000 students in a university. Can we say with certainty that there are 12 students with the same birthday? Explain.

Solution: NO.
$366 \times 11 = 4026 > 4000$. Therefore it is possible to distribute 4000 people to 366 groups such that each group contains 11 or fewer people.

Exercise 7-11: Let $A = \{3, 6, 9, \ldots, 99\}$ and $B = \{2, 4, 6, \ldots, 100\}$.

a) How many functions are there from A to B?

b) How many one-to-one functions are there from A to B?

c) How many functions are there from A to B that take multiples of 10 to multiples of 10? (For example, $f(30) = 20, f(60) = 60$ etc.)

Solution: **a)** 50^{33}

b) $\dfrac{50!}{(50-33)!} = 50 \cdot 49 \cdots 18$

c) $10^3\, 50^{30}$

Exercise 7-12: Assume a country has a population of 74 million. There are 100 cities and 20 football teams in this country. Everyone lives in a city and supports one team. We can certainly find 2 or 3 people who have the same birthday, live in the same city and support the same team.

What is the maximum number of such people that we can always find?

Solution: There are 74 million people and $366 \times 100 \times 20 = 736\,000$ categories. If we distribute all people there will be

$$\frac{74\,000\,000}{736\,000} = 101.09$$

people per category on average.

$$\lceil 101.09 \rceil = 102$$

By pigeonhole principle, at least one category will certainly contain 102 people.

Week 8– Inclusion and Exclusion

The Principle of Inclusion and Exclusion: Consider a set S with $|S| = N$ and conditions $c_1, c_2, \ldots, c_m$ on the elements of S. The number of elements of S that satisfy none of the conditions c_i is given by:

$$\begin{aligned}\overline{N(c_1 c_2 \ldots c_m)} = \;& N \\ & -[N(c_1) + \cdots + N(c_m)] \\ & +[N(c_1c_2) + \cdots + N(c_{m-1}c_m)] \\ & -[N(c_1c_2c_3) + \cdots] \\ & \quad\vdots \\ & (-1)^m N(c_1c_2 \ldots c_m)\end{aligned}$$

Proof: Consider an element that satisfies r of these conditions. then we are adding it

$$1 - r + \binom{r}{2} - \binom{r}{3} + \cdots (-1)^r \binom{r}{r} = \big[1 + (-1)\big]^r$$

times. That is, 0 times.

Exercise 8-1: Determine the number of positive integers n where $1 \leqslant n \leqslant 100$ and n is not divisible by $2, 3$ or 5.

$\Bigg($**Answer:** $100 - 50 - 33 - 20 + 16 + 10 + 6 - 3 = 26\Bigg)$

Number of Elements in Set Intersections

How can we find the number of elements in $A_1 \cup A_2 \cup \cdots \cup A_n$?

Add all the sets, then subtract intersections of two sets, then add intersections of three sets etc.

Exercise 8-2: Find the number of integers in $\{1, 2, \ldots, 1000\}$ that are divisible by 6,7 or 9.

$$\begin{aligned}|D_6 \cup D_7 \cup D_9| &= |D_6| + |D_7| + |D_9| - |D_6 \cap D_7| - |D_6 \cap D_9| - |D_7 \cap D_9| + |D_6 \cap D_7 \cap D_9| \\ &= |D_6| + |D_7| + |D_9| - |D_{42}| - |D_{18}| - |D_{63}| + |D_{126}|\end{aligned}$$

Exercise 8-3: In how many ways can we divide 12 students into 3 sets with one set having 2 and others having 5 students?

$\Bigg($**Answer:** $8316\Bigg)$

Exercise 8-4: In how many different ways can we factorize 156009, where each factor is greater than 1 and order is not important?(Hint: $156009 = 3 \cdot 7 \cdot 17 \cdot 19 \cdot 23$)

$\left(\textbf{Answer: } 51 = 15 + 25 + 10 + 1\right)$

Exercise 8-5: In how many ways can three x's, three y's and three z's be arranged so that no consecutive triplet of the same letter appears?

$\left(\textbf{Answer: } \dfrac{9!}{3!\,3!\,3!} - 3\,\dfrac{7!}{3!\,3!} + 3\,\dfrac{5!}{3!} - 3! = 1314\right)$

Exercise 8-6: How many numbers from the set $S = \{1000, 1001, \ldots, 9999\}$ have at least one digit 0, at least one 8 and at least one 9?

$\left(\textbf{Answer: } 9000 - 9^4 - 8 \cdot 9^3 - 8 \cdot 9^3 + 8^4 + 8^4 + 7 \cdot 8^3 - 7^4 = 150\right)$

Exercise 8-7: In how many different ways can we distribute 15 identical marbles to 5 different children such that each child has at least one and at most 4 marbles?

$\left(\textbf{Answer: } \binom{14}{10} - \binom{5}{1}\binom{10}{6} + \binom{5}{2}\binom{6}{2} = 101\right)$

Exercise 8-8: On a graph, we will connect 5 distinct vertices with 4 edges such that no vertex will be isolated. In how many different ways can we do this?

$\left(\textbf{Answer: } 2^{10} - \binom{5}{1} 2^6 + \binom{5}{2} 2^3 - \binom{5}{3} 2^1 + \binom{5}{4} 2^0 - \binom{5}{5} 2^0 = 768\right)$

Exercise 8-9: Six married couples will be seated at a circular table. In how many ways can we do this such that no wife sits next to her husband?

$\left(\textbf{Answer: } 12\,771\,840\right)$

DERANGEMENTS

In a derangement, nothing is in its natural place.

Exercise 8-10: Find the number of permutations of n such that no number is in its correct place for $n = 2, 3, 4, 5$.

$\left(\textbf{Answer: } 1, 2, 9, 44\right)$

Exercise 8-11: A secretary mails 10 letters without checking the envelopes for correct address. What is the probability that all letters are sent to wrong locations?

$$\left(\textbf{Answer: } \left[10! - \binom{10}{1}9! + \binom{10}{2}8! + \cdots + \binom{10}{10}0!\right]/10! \approx \frac{1}{e}\right)$$

Exercise 8-12: If we distribute 9 exam papers to 9 students randomly, what is the probability that none of them receives his/her own paper?

$$9! - 9 \cdot 8! + \binom{9}{2} \cdot 7! - \binom{9}{3} \cdot 6! + \cdots + \binom{9}{8} \cdot 1! - \binom{9}{9} \cdot 0!$$

If we divide this by the total number of distributions 9!, we obtain

$$1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{1}{8!} - \frac{1}{9!} \approx \frac{1}{e}$$

The case where at least one gets correct paper has probability $\approx 1 - \frac{1}{e}$

STIRLING NUMBERS

The number of different ways to distribute m distinct objects to n identical containers, with no container left empty is $S(m,n)$

$m \setminus n$	1	2	3	4	5	6
1	1					
2	1	1				
3	1	3	1			
4	1	7	6	1		
5	1	15	25	10	1	
6	1	31	90	65	15	1

If we construct this table's a few rows carefully, we will notice that

$$S(m,n) = S(m-1,n-1) + nS(m-1,n)$$

This can be solved to give:

$$S(m,n) = \frac{1}{n!}\sum_{k=0}^{n}(-1)^k\binom{n}{n-k}(n-k)^m$$

Exercise 8-13: How many onto functions are there from A to B?

For $|A| = m,\ |B| = n$, there are $n!S(m,n)$ onto functions from A to B. (If $m \geqslant n$).

Another way to express the same formula is:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{n-k}(n-k)^m$$

Exercise 8-14: In how many ways can we distribute 7 distinct problems to 4 distinct students?

$\left(\textbf{Answer: } \binom{4}{4}4^7 - \binom{4}{3}3^7 + \binom{4}{2}2^7 - \binom{4}{1}1^7 = 8400\right)$

Exercise 8-15: How many numbers from the set $S = \{1, 2, \ldots, 1000\}$ are divisible by at least one of the numbers 3, 5 or 11?

$\left(\textbf{Answer: } 333 + 200 + 90 - 66 - 30 - 18 + 6 = 515\right)$

Exercise 8-16: Consider all strings of 4 letters made of letters $\{v, w, x, y, z\}$. Some examples are $xyvw, xxyy, zyvz$. How many of them contain at least one x, at least one y and at least one z?

$\left(\textbf{Answer: } 5^4 - 3 \cdot 4^4 + 3 \cdot 3^4 - 2^4 = 84\right)$

Exercise 8-17: Turkish alphabet has 29 letters, $\{A, B, C, \ldots, Z\}$. Clearly, there are 29! permutations of this set. How many of these permutations do not contain any of the words AYŞE, MERT, NUR or CANSU?

Solution: The number of permutations is:

29!	(All permutations)
−26!	(Contains AYŞE)
−26!	(Contains MERT)
−27!	(Contains NUR)
−25!	(Contains CANSU)
+24!	(Contains both AYŞE and NUR)
+22!	(Contains both MERT and CANSU)

Exercise 8-18: 8 students, Nilay, Özgür, Arda, Cansu, Kübra, Merve, Mert and Gizem will sit around a circular table.

- Arda and Mert want to sit next to each other.
- Merve and Nilay do NOT want to sit next to each other.
- Cansu and Kübra do NOT want to sit next to each other.
- Merve and Cansu do NOT want to sit next to each other.

In how many different ways can they be seated?

Solution: Consider Arda and Mert as a single person. Thus, there are 7 persons, but this is a circular table, so there are a total of 6! arrangements.

Now we have to subtract cases where M-N, C-K or M-C are together, add the cases both M-N and C-K, or both M-N and M-C or both C-K and M-C are together, (be careful at this point, M-N+C-K is two couples, M-N+M-C is a triple) and then subtract cases where all three are together.

Then, we have to multiply by 2 because Arda and Mert can exchange places. So the answer is:

$$2 \cdot \left(6! - 2 \cdot 5! - 2 \cdot 5! - 2 \cdot 5! + 4 \cdot 4! + 2 \cdot 4! + 2 \cdot 4! - 2 \cdot 3!\right) = 360$$

Exercise 8-19: There are 9 students in a class, Ayça, Batuhan, Çağatay, Damla, Elif, Turaç, Uğur, Volkan, Zehra. We will form 4 project groups from these students. Each group may contain at least 1 and at most 6 students. In how many different ways can we do this if

a) Projects are identical?

b) Projects are distinct?

Solution: We can think of this problem as the number of onto functions from a 9 element set to a 4 element set.

Extend the Stirling table up to 9th level using the formula
$S(m, n) = S(m-1, n-1) + nS(m-1, n)$:

$m \setminus n$	1	2	3	4	5	6	
1	1						
2	1	1					
3	1	3	1				
4	1	7	6	1			
5	1	15	25	10	1		
6	1	31	90	65	15	1	
7	1	63	301	350	140	21	1
8	1	127	966	1701	1050	$\cdots$	
9	1	255	3025	7770	$\cdots$		

So:

a) 7770

b) $4!\,7770 = 186\,480$

Second Method:

We can distribute students to subsets as follows:

$$1+2+3+3: \quad 9\binom{8}{2}\binom{6}{3}/2 = 2520 \quad \text{ways}$$

$$2+2+2+3: \quad \binom{9}{2}\binom{7}{2}\binom{5}{2}/3! = 1260 \quad \text{ways}$$

$$1+1+3+4: \quad \binom{9}{4}\binom{5}{3} = 1260 \quad \text{ways}$$

$$1+2+2+4: \quad \binom{9}{4}\binom{5}{2}\binom{3}{2}/2 = 1890 \quad \text{ways}$$

$$1+1+2+5: \quad \binom{9}{5}\binom{4}{2} = 756 \quad \text{ways}$$

$$1+1+1+6: \quad \binom{9}{6} = 84 \quad \text{ways}$$

Total is: $2520 + 1260 + 1260 + 1890 + 756 + 84 = 7770$ different ways.

Exercise 8-20: 6 students, A, B, C, D, E, F go to a cafe where there are 3 empty tables. They take seats such that no table is left empty. In how many different ways can they do this? (Tables are identical)

Solution:

$$\frac{3^6 - \binom{3}{1}2^6 + \binom{3}{2}1^6}{3!} = 90$$

Second Method: We can distribute 6 elements into 3 sets as $1-1-4$, $\quad 1-2-3$, or $2-2-2$

$$\binom{6}{4} + \binom{6}{3}\binom{3}{2} + \frac{\binom{6}{2}\binom{4}{2}}{3!} = 90$$

Exercise 8-21: 5 students, A, B, C, D, E go to a cafe where there are 3 empty tables. One is near the cashier, the second is in the middle and the third is near the window. They take seats such that no table is left empty. In how many different ways can they do this?

Solution:

$$3^5 - 3 \cdot 2^5 + 3 \cdot 1^5 = 150$$

Second Method: We can distribute 5 elements into 3 sets as $1-1-3$ or $1-2-2$

$$\binom{5}{3} + \frac{\binom{5}{2}\binom{3}{2}}{2!} = 25$$

Now we have to multiply this by 3! because tables are distinct.

$$25 \cdot 3! = 150$$

Exercise 8-22: How many positive integers n less than 900 satisfy $\gcd(n, 900) = 1$?

Solution:

$$900 - \frac{900}{2} - \frac{900}{3} - \frac{900}{5} + \frac{900}{2 \cdot 3} + \frac{900}{2 \cdot 5} + \frac{900}{3 \cdot 5} - \frac{900}{2 \cdot 3 \cdot 5}$$
$$= 240$$

Exercise 8-23: How many words of seven letters composed of the set $\{a, b, c, d, e\}$ contain all five symbols?

Solution:

$$5^7 - \binom{5}{1}4^7 + \binom{5}{2}3^7 - \binom{5}{3}2^7 + \binom{5}{4}1^7$$

$$= 16800$$

OR

$$\{a, b, c, d, e\} + \text{ two identical letters, for example } a, a: \binom{5}{1}\frac{7!}{3!} = 4200$$

$$\{a, b, c, d, e\} + \text{ two distinct letters, for example } a, b: \binom{5}{2}\frac{7!}{2!2!} = 12600$$

$$4200 + 12600 = 16800$$

Week 9– Generating Functions

Formal Power Series:

$$\sum_{n=0}^{N} x^n = 1 + x + x^2 + x^3 + \cdots + x^N = \frac{1 - x^{N+1}}{1 - x}$$

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1 - x}$$

$$\sum_{n=1}^{\infty} x^n = x + x^2 + x^3 + \cdots = \frac{x}{1 - x}$$

$$\sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \cdots = \frac{1}{1 + x}$$

$$\sum_{n=0}^{\infty} x^{2n} = 1 + x^2 + x^4 + x^6 + \cdots = \frac{1}{1 - x^2}$$

$$\sum_{n=0}^{\infty} (n+1) x^n = 1 + 2x + 3x^2 + \cdots = \frac{1}{(1 - x)^2}$$

$$\frac{1}{2}\sum_{n=0}^{\infty} (n+2)(n+1) x^n = 1 + 3x + 6x^2 + \cdots = \frac{1}{(1 - x)^3}$$

If we generalize this by evaluating derivatives $m - 1$ times, we obtain

$$\frac{1}{(1-x)^m} = \sum_{n=0}^{\infty} \binom{m+n-1}{n} x^n$$

$$= 1 + mx + \frac{m(m+1)}{2} x^2 + \frac{m(m+1)(m+2)}{3!} x^3 + \frac{m(m+1)(m+2)(m+3)}{4!} x^4 + \cdots$$

We are not interested in convergence here, but the above series converge for $|x| < 1$.

Exercise 9-1: What is the coefficient of x^{10} in the expansion of $(x + x^2)(x^3 + x^4 + \cdots)(1 + x^2 + x^4 + \cdots)$?

$\left(\textbf{Answer: } 7\right)$

Exercise 9-2: In how many ways can we distribute 10 identical balls to three different children A, B, C such that A gets 1 or 2, B gets 3 or more, C gets an even number of balls?

$\left(\textbf{Answer: } 7, \text{ The problems are equivalent}\right)$

Exercise 9-3: In how many ways can we distribute 12 oranges to A, B, C such that A gets at least 4, B gets at least 2 and C gets between 2 and 5?

$\left(\textbf{Answer: } 14\right)$

Exercise 9-4: Determine the coefficient of x^{15} in $(x^2 + x^3 + \cdots)^4$.

$\left(\textbf{Answer: } 120\right)$

Exercise 9-5: Determine the coefficient of x^{10} in the expansion of $\dfrac{x}{(1+x^2)(1-x)}$

Exercise 9-6: In how many ways can we fill a bag with n fruits subject to the following constraints?

- The number of apples must be even.
- The number of bananas must be a multiple of 5.
- There can be at most four oranges.
- There can be at most one pear.

For example, there are 8 ways to form a bag with 7 fruits:

Apples	6	6	4	4	2	2	0	0
Bananas	0	0	0	0	0	5	5	5
Oranges	0	1	3	2	4	0	2	1
Pears	1	0	0	1	1	0	0	1

$\left(\textbf{Answer: } \dfrac{1}{1-x^2}\,\dfrac{1}{1-x^5}\,\dfrac{1-x^5}{1-x}\,(1+x) = \dfrac{1}{(1-x)^2} = 1+2x+3x^2+\cdots\right)$

So the answer is simply $n+1$.

Exercise 9-7: In how many different ways can we distribute n apples to A, B, C such that A gets 0 or 1, B gets 0 or 2, C gets a multiple of 4?

$\left(\textbf{Answer: } 1. \text{ The result is independent of } n\right)$

Exercise 9-8: In how many different ways can we distribute 75 pieces of chocolate to 5 different children such that each child gets at least 6 and at most 24?

Solution: The generating function is:

$$\begin{aligned}(x^6+x^7+\cdots+x^{24})^5 &= x^{30}(1+x+x^2+\cdots+x^{18})^5 \\ &= x^{30}\,\frac{(1-x^{19})^5}{(1-x)^5}\end{aligned}$$

$$= x^{30}(1-5x^{19}+10x^{38}-10x^{57}+5x^{76}-x^{95})\sum_{n=0}^{\infty}\binom{n+4}{n}x^n$$

Coefficient of x^{75} is: $\dbinom{49}{45} - 5\dbinom{30}{26} + 10\dbinom{11}{7} = 78151$

Second Method: Give 6 to each, distribute remaining 45 balls randomly. Subtract cases where one child gets 19 or more. Add cases where two children gets 19 or more, to obtain:
$\dbinom{49}{45} - 5\dbinom{30}{26} + 10\dbinom{11}{7} = 78151$

Exercise 9-9: We are distributing 33 pieces of identical bonibons to Nilay, Ahmet Can, Mehmet, Cihan and Büşra with the following constraints:

- Nilay gets a multiple of 6. $\{0, 6, 12, \ldots\}$
- Ahmet Can gets at most 5. $\{0, 1, \ldots, 5\}$
- Mehmet gets a multiple of 4. $\{0, 4, 8, \ldots\}$
- Cihan gets none or 2. $\{0, 2\}$
- Büşra gets 7 or 8. $\{7, 8\}$

In how many different ways can we do this distribution?

Solution: The generating function is:

$$(1 + x^6 + x^{12} + \cdots)(1 + x + \cdots + x^5)(1 + x^4 + x^8 + \cdots)(1 + x^2)(x^7 + x^8)$$

$$\begin{aligned}
&= \frac{1}{1-x^6}\,\frac{1-x^6}{1-x}\,\frac{1}{1-x^4}\,(1+x^2)x^7(1+x) \\
&= \frac{x^7(1+x)(1+x^2)}{(1-x)(1+x^2)(1-x)(1+x)} \\
&= \frac{x^7}{(1-x)^2} \\
&= x^7(1 + 2x + 3x^2 + 4x^3 + \cdots)
\end{aligned}$$

Coefficient of x^{33} is $33 - 6 = 27$. Therefore there are 27 different ways.

Exercise 9-10: We will buy a total of 100 identical laptop computers from 5 distinct stores A, B, C, D, E such that

- We will buy an odd number from A,
- We will buy 0 or 1 from B,
- We will buy 10 or less from C,
- We will buy 7 or more from D.

In how many different ways can we do that?

Solution: The generating function is:

$$(x + x^3 + x^5 + \cdots)(1 + x)(1 + x + \cdots + x^{10})(x^7 + x^8 + \cdots)(1 + x + x^2 + \cdots)$$

$$\begin{aligned}
&= \frac{x}{1-x^2}\,(1+x)\,\frac{1-x^{11}}{1-x}\,\frac{x^7}{1-x}\,\frac{1}{1-x} \\
&= \frac{x^8 - x^{19}}{(1-x)^4} \\
&= (x^8 - x^{19}) \sum_{n=0}^{\infty} \binom{n+3}{n} x^n
\end{aligned}$$

Coefficient of x^{100} is:

$$\binom{95}{92} - \binom{84}{81} = 43\,131$$

Exercise 9-11: Find the number of different solutions of the equation $n + m + k = 10$ where n, m, k are nonnegative integers and n is odd, m is 0 or 1, k is at most 8.

Solution:

$$\begin{aligned}
&(x + x^3 + x^5 + \cdots)(1 + x)(1 + x + x^2 + \cdots + x^8) \\
&= \frac{x}{1-x^2}(1+x)\frac{1-x^9}{1-x} \\
&= \frac{x(1-x^9)}{(1-x)^2} \\
&= (x - x^{10})\sum_{n=0}^{\infty}\binom{n+1}{n}x^n
\end{aligned}$$

Coefficient of x^{10}:

$$\binom{10}{9} - \binom{1}{0} = 9$$

Exercise 9-12: We will order 6 pizzas. There are 3 types available: Vegetarian, pepperoni and margarita. We want at most 3 vegetarian, at most 4 margarita and at least 2 pepperoni. In how many different ways can we order?

Solution:

$$\begin{aligned}
&(1 + x + x^2 + x^3)(1 + x + x^2 + x^3 + x^4)(x^2 + x^3 + \cdots) \\
&= \frac{1-x^4}{1-x}\frac{1-x^5}{1-x}\frac{x^2}{1-x} \\
&= \frac{x^2 - x^6 - x^7 + x^{11}}{(1-x)^3} \\
&= (x^2 - x^6 - x^7 + x^{11})\sum_{n=0}^{\infty}\binom{n+2}{n}x^n
\end{aligned}$$

Coefficient of x^6:

$$\binom{6}{4} - \binom{2}{0} = 14$$

Week 10– Partitions of Integers

Let n be a positive integer. The number of ways we can express n as a sum of positive integers, where order is NOT important, is called $p(n)$. For example

$$\begin{aligned} p(3) &= 3, \quad (3,\ 1+2,\ 1+1+1) \\ p(4) &= 5 \\ p(5) &= 7 \end{aligned}$$

Question: What is $p(8)$?

It is the coefficient of x^8 in the expansion of:

$$(1+x+x^2+\cdots)(1+x^2+x^4+\cdots)(1+x^3+x^6+\cdots)\cdots(1+x^8+x^{16}+\cdots)$$
$$= \frac{1}{1-x}\,\frac{1}{1-x^2}\,\frac{1}{1-x^3}\cdots\frac{1}{1-x^8}$$

Note that the result wouldn't change if we expressed the product as:

$$(1+x+x^2+\cdots+x^8)(1+x^2+x^4+x^6+x^8)(1+x^3+x^6)\cdots(1+x^8)$$

Exercise 10-1: Find the generating function for the partitions of 11 where components are distinct.

$\left(\textbf{Answer: } (1+x)(1+x^2)\cdots(1+x^{11})\right)$

Exercise 10-2: Find the generating function for the number of partitions of n where each component is an odd number.

$\left(\textbf{Answer: } \frac{1}{1-x}\,\frac{1}{1-x^3}\,\frac{1}{1-x^5}\cdots\right)$

Exercise 10-3: Find the generating function for the number of partitions of n where each component is an odd number, and it is repeated an odd number of times.

$\left(\textbf{Answer: } (1+x+x^3+x^5+\cdots)(1+x^3+x^9+x^{15}+\cdots)(1+x^5+x^{15}+x^{25}+\cdots)\cdots\right)$

Exercise 10-4: Find the generating function for the number of integer solutions of $3p+5q+12r=n$ where $0 \leqslant p, q, r$.

$\left(\textbf{Answer: } \frac{1}{1-x^3}\,\frac{1}{1-x^5}\,\frac{1}{1-x^{12}}\right)$

EXPONENTIAL GENERATING FUNCTIONS

For a sequence of real numbers $a_0, a_1, a_2, \dots$ the exponential generating function is defined as:

$$f(x) = a_0 + a_1 x + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \cdots$$

The generating function for the sequence $1, 1, 1, \dots$ is e^x.

For even powers, we can use $\dfrac{e^x + e^{-x}}{2}$ and for odd powers, $\dfrac{e^x - e^{-x}}{2}$.

Exercise 10-5: In how many ways can four of the letters of the word ENGINE be arranged? (Here, order is important)

$\left(\textbf{Answer: } 102\right)$

We can obtain the same result by finding the coefficient of $x^4/4!$ in the expansion of:

$$\left[1 + x + \frac{x^2}{2!}\right]^2 (1+x)^2$$

Exercise 10-6: A ship carries 48 flags, 12 each of the colors red, white, blue, black. Twelve of these are placed on a vertical pole.

a) How many such signals use an even number of blue flags and an odd number of black flags?

$\left(\textbf{Answer: } 4^{11}\right)$

b) How many of the signals have at least three white flags or no white flags at all?

$\left(\textbf{Answer: } 4^{12} - 12 \cdot 3^{11} - 66 \cdot 3^{10}\right)$

Exercise 10-7: Consider all 50 letter sequences made of letters a, b, c.

a) How many of them contain an even number of a's?

$\left(\textbf{Answer: } \dfrac{3^{50}+1}{2}\right)$

b) How many of them contain odd number of b's?

$\left(\textbf{Answer: } \dfrac{3^{50}-1}{2}\right)$

Exercise 10-8: A set has 10 elements. We will partition it into 4 subsets such that each subset will have an odd number of elements. In how many different ways can we do this?

$\left(\textbf{Answer: } 5440, \text{ Use generating function: } \dfrac{1}{4!}\left(\dfrac{e^x - e^{-x}}{2}\right)^4\right)$

Exercise 10-9: There are 32 houses on a street. We will paint each house with one of the six colors: White, blue, red, yellow, green or orange. For each color, the number of houses painted must be even.

In how many different ways can we do that?

Solution: The exponential generating function is:

$$\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)^6 = \left(\frac{e^x+e^{-x}}{2}\right)^6$$

$$= \frac{e^{6x}+6e^{4x}+15e^{2x}+20+15e^{-2x}+6e^{-4x}+e^{-6x}}{64}$$

Coefficient of $\dfrac{x^{32}}{32!}$ is:

$$\frac{1}{32}\left(6^{32}+6\cdot 4^{32}+15\cdot 2^{32}\right)$$

Exercise 10-10: Consider all words of length 10 made of the letters in the set $\{a, b, c, d, e, f, g\}$. How many contain at least one a, at least one b and at least two c?

Solution: Define conditions as:

- P: The word contains no a,
- Q: The word contains no b,
- R: The word contains no c or 1 c.

Then, the answer is:

$$\begin{aligned} &= N - N(P) - N(Q) - N(R) \\ &\quad +N(PQ) + N(PR) + N(QR) \\ &\quad -N(PQR) \end{aligned}$$

$$\begin{aligned} &= 7^{10} - 6^{10} - 6^{10} - (6^{10} + 10 \cdot 6^9) \\ &\quad +5^{10} + (5^{10} + 10 \cdot 5^9) + (5^{10} + 10 \cdot 5^9) \\ &\quad -(4^{10} + 10 \cdot 4^9) \end{aligned}$$

Second Method: The exponential generating function is:

$$\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)^2 \left(\frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)^4$$

$= (e^x - 1)^2(e^x - 1 - x)e^{4x}$

$= e^{7x} - (3 + x)e^{6x} + (3 + 2x)e^{5x} - (1 + x)e^{4x}$

The coefficient of $\dfrac{x^{10}}{10!}$ is:

$7^{10} - 3 \cdot 6^{10} - 10 \cdot 6^9 + 3 \cdot 5^{10} + 20 \cdot 5^9 - 4^{10} - 10 \cdot 4^9$

Week 11– Computational Complexity

Growth of Functions: Given two increasing functions, how can we tell which one is increasing faster?

Big-Oh: Let $f(n), g(n)$ be two functions. We say $f(n) = O(g(n))$ (f is Big-Oh of g) if

$$0 \leqslant f(n) \leqslant cg(n)$$

for all $n \geqslant n_0$. In other words, $f(n) = O(g(n))$ if $g(n)$ times a constant is greater than $f(n)$ for sufficiently large n values.

Exercise 11-1: Show that $n^k = O(e^n)$ for all $k \in \mathbb{Z}^+$.

Exercise 11-2: Show that $\log n = O(n^k)$ $k > 0$.

Exercise 11-3: Show that $2^n = O(n!)$

Big-Theta: We say that $f(n) = \Theta(g(n))$ (f is Big-theta of g) if

$$0 \leqslant c_1 g(n) \leqslant f(n) \leqslant c_2 g(n)$$

for all $n \geqslant n_0$. In other words, if $f(n) = O(g(n))$ and $g(n) = O(f(n))$ then we say $f(n) = \Theta(g(n))$.

Exercise 11-4: Show that $(n+1)^2 = \Theta(n^2)$

Exercise 11-5: Show that $\ln n = \Theta(\log n)$

Exercise 11-6: Determine the Θ class of the following:

a) $f(n) = n^2 - 2^n - \ln n$

b) $f(n) = 3(n-1)^3$

c) $f(n) = n! - e^n$

$\Big($**Answer:**$\Big)$ **a)** $\Theta(2^n)$

b) $\Theta(n^3)$

c) $\Theta(n!)$

Exercise 11-7: Rank the following functions according to their growth rates:

$$n, n!, 2^n, n^2, n^3, \log n, n \log n, \sqrt{n}, n\sqrt{n}, 3^n, n^n, 1$$

Exercise 11-8: Rank the following functions according to their growth rates:

$$f(n) = n^2 + 2n\sqrt{n}, \quad g(n) = n^2 \log n - n \log n, \quad h(n) = 1 + 2n^{3/2} + \left(\frac{3}{2}\right)^n$$

Solution: $f = O(g),\ g = O(h)$. In other words for sufficiently large n we have $f < g < h$.

Exercise 11-9: Rank the following functions according to their growth rates:

$$n + n^2, \quad n! + e^n, \quad n\sqrt{n}, \quad 4^n, \quad (n+1)^n$$

Solution:

$$n\sqrt{n} < n + n^2 < 4^n < n! < (n+1)^n$$

Exercise 11-10: Rank the following functions according to their growth rates:

$$e^{(n+2)}, \quad n^n + n!, \quad \log(n^2), \quad n \log n, \quad (2n)!$$

Solution:

$$\log(n^2) < n \log n < e^{(n+2)} < n^n < (2n)!$$

For the following questions, write an algorithm in pseudo-code. Then, find the asymptotic complexity of the algorithm:

Exercise 11-11: Given an unsorted vector a, find if the element X occurs in a.

Solution:

```
    INPUT vector a
n = size(a)
For i = 1 to n
      If a(i) == X
           Return True;
      EndIf
EndFor
Return False;
```

$\left(\textbf{Answer: } O(n)\right)$

Exercise 11-12: Given a sorted vector a, find the location of element X

Solution:

```
    INPUT vector a
low = 0
high = size(a)
While low ⩽ high
     mid = (low + high)/2
     If a(mid) < X
          low = mid + 1
     elseif a(mid) > X
          high = mid − 1
     else
          Return mid;
     EndIf
EndWhile
Return NOT FOUND;
```

$\left(\textbf{Answer: } O(\ln(n))\right)$

Exercise 11-13: Sort N numbers by finding the smallest in the list and placing it as the first:

Solution:

```
    For i = 1 to N
     index = i
     For j = i + 1 to N
          If A(j) < A(index)
               index = j
          EndIf
     EndFor
     Swap(A(i), A(index))
EndFor
```

Exercise 11-14: Given an array of n elements, check whether all elements are distinct. Return False if any two are the same.

Solution:

```
INPUT: Vector A
n = size(A)
For i = 1 to n − 1
    For j = i + 1 to n
        If A(i) == A(j)
            Return False
        EndIf
    EndFor
EndFor
Return True
```

Exercise 11-15: Find the time the following algorithm takes based on input size n. In other words, find Θ class. You may assume n is a power of 2 for simplicity.

Solution:

```
INPUT n
S = 0; k = 1
While k ⩽ n
    For j = 1 to n
        S = S + 1
    EndFor
    k = 2 ∗ k
EndWhile
```

$\left(\textbf{Answer: } \Theta(n \log_2 n)\right)$

Exercise 11-16: Find the time the following algorithm takes based on input size n. In other words, find Θ class. You may assume n is a power of 2 for simplicity.

Solution:

```
INPUT n
S = 0; k = 1
While k ⩽ n
    For j = 1 to k
        S = S + 1
    EndFor
    k = 2 ∗ k
EndWhile
```

$\left(\textbf{Answer: } \Theta(n)\right)$

Exercise 11-17: The following program checks if all elements of the given vector are distinct. It returns False if any two are the same. What is the asymptotic growth rate? (Consider worst case)

Solution:

```
  INPUT: Vector A
n = size(A)
For i = 1 to n − 1
    For j = i + 1 to n
        If A(i) == A(j)
            Return False
        EndIf
    EndFor
EndFor
Return True
```

Solution: $(n-1)+(n-2)+(n-3)+\cdots+2+1=\dfrac{(n-1)n}{2}=\Theta(n^2)$

Exercise 11-18: Find the time the following algorithm takes based on input size n. In other words, find its Θ class.

Solution:

```
  INPUT n
Sum = 0
For i = 1 to n
    For j = 1 to i^2
        Sum = Sum + j
    EndFor
EndFor
```

Solution: $1^2+2^2+3^2+\cdots+n^2=\dfrac{n(n+1)(2n+1)}{6}=\dfrac{2n^3+3n^2+n}{6}$

$\Rightarrow$ Complexity of Algorithm is: $\Theta(n^3)$

Exercise 11-19: Find a function $g(n)$ such that $f(n) = O(g(n))$ and $g(n) = O(h(n))$ but $f(n) \neq \Theta(g(n))$ and $g(n) \neq \Theta(h(n))$.

In other words, $g(n)$ should increase faster that $f(n)$ but slower than $h(n)$. For example: $f(n) = n^2, \quad h(n) = n^4 \quad \Rightarrow \quad g(n) = n^3$

a) $f(n) = \ln n, \quad h(n) = n$

b) $f(n) = 2^n, \quad h(n) = n^n$

c) $f(n) = n^2 + n, \quad h(n) = n^2 + e^n$

Solution: Many answers are possible, some examples are:

a) $\sqrt{n}$

b) $n!$ or 3^n or $n2^n$

c) n^3 or n^4 or 2^n

Exercise 11-20: We know that `Fonk1(n)` is an algorithm of complexity $\Theta(n)$ and `Fonk2(n)` is an algorithm of complexity $\Theta(n^2)$. Find the complexity of the following algorithm:

INPUT n
For $i = 1$ to $3n$
    Fonk1(i)
    Fonk2(i)
EndFor
For $j = 1$ to n
    Fonk1(n)
EndFor

Solution:

$$\begin{aligned}\Big(1 + 2 + \cdots + 3n\Big) + \Big(1^2 + 2^2 + \cdots + (3n)^2\Big) + \Big(n + \cdots + n\Big) &= \Theta(n^2) + \Theta(n^3) + \Theta(n^2) \\ &= \Theta(n^3)\end{aligned}$$

Week 12– Homogeneous Recurrence Relations

We say that a sequence is defined recursively if some finite set of values, usually the first few are specified, and the remaining values are defined in terms of the previous ones.

For example, the factorial function defined as: Factorial(0) = 1,
Factorial(n) = Factorial($n-1$) is recursive.

Fibonacci sequence is another example.

Exercise 12-1: Solve the first order recurrence relation $a_{n+1} = 2a_n, \quad a_0 = 7.$

$\left(\textbf{Answer: } 7 \cdot 2^n\right)$

Exercise 12-2: Solve $a_n - 3a_{n-1} = 0, \quad a_4 = 81$

$\left(\textbf{Answer: } 3^n\right)$

Exercise 12-3: Find an explicit formula for a_n where $a_1 = 1,\ a_2 = 5$ and $a_{n+1} = 5a_n - 6a_{n-1}$ for $n \geqslant 2$.

$\left(\textbf{Answer: } a_n = 3^n - 2^n\right)$

Assume the result is x^n like the first order equations. We obtain $x^2 - 5x + 6 = 0$. The solution is 2 or 3, and both 2^n and 3^n satisfy the equation. Also, $2^n + 3^n$ or any linear combination of these two satisfy the equation.

The following theorem summarizes these results:

Theorem: Let $a_n + pa_{n-1} + qa_{n-2} = 0$ be a recurrence relation and $x^2 + px + q = 0$ its characteristic equation, where p, q are nonzero.

a) If the characteristic equation has two distinct roots r_1, r_2 then $a_n = c_1 r_1^n + c_2 r_2^n$.

b) If the characteristic equation has a double root r then $a_n = c_1 r^n + c_2 n r^n$.

Exercise 12-4: Solve the recurrence relation $a_0 = 1,\ a_1 = -3$ and $a_n = -2a_{n-1} + 3a_{n-2}$ for $n \geqslant 2$.

$\left(\textbf{Answer: } a_n = (-3)^n\right)$

Exercise 12-5: How can we obtain the general formula for Fibonacci sequence, given that $F_n = F_{n-1} + F_{n-2}$?

$$\left(\textbf{Answer: } \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]\right)$$

Assume $F_n = x^n$. In that case, $x^n = x^{n-1} + x^{n-2}$ and $x^2 - x - 1 = 0$. The solution of this equation gives $x_{1,2} = \dfrac{1 \pm \sqrt{5}}{2}$, therefore the general solution is $F_n = c_1 x_1^n + c_2 x_2^n$. We can determine c_1 and c_2 from the initial conditions $F_1 = 1, F_2 = 1$.

Exercise 12-6: Solve $a_n + a_{n-1} - 6a_{n-2} = 0, n \geqslant 2, \quad a_0 = -1, a_1 = 8$

$\left(\textbf{Answer: } a_n = 2^n - 2(-3)^n\right).$

Exercise 12-7: Solve $a_{n+2} = a_{n+1} + a_n, a_0 = 0, a_1 = 4$.

Exercise 12-8: (Repeated Root) Solve the recurrence relation $a_{n+2} = 4a_{n+1} - 4a_n$, where $a_0 = 1, a_1 = 3$

$\Bigg($**Answer:** $a_n = 2^n + n2^{n-1}\Bigg)$

Exercise 12-9: (Repeated Root) Solve the recurrence relation $9a_{n+2} = 42a_{n+1} - 49a_n$, where $a_1 = 21, a_2 = 98$

$\Bigg($**Answer:** $a_n = 9n\left(\frac{7}{3}\right)^n\Bigg)$

Exercise 12-10: Solve $2a_{n+3} = a_{n+2} + 2a_{n+1} - a_n, \quad a_0 = 0, a_1 = 1, a_2 = 2$

$\Bigg($**Answer:** $a_n = \frac{5}{2} + \frac{1}{6}(-1)^n - \frac{8}{3}\left(\frac{1}{2}\right)^n\Bigg)$

Complex Roots

For the following questions, remember the formulas:

$$x + iy = re^{i\theta}, \quad \text{where } r = \sqrt{x^2 + y^2}, \quad \text{and } \theta = \arctan\frac{y}{x}$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\Rightarrow \quad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Exercise 12-11: Solve the recurrence relation $a_{n+2} = 2a_{n+1} - 2a_n$, where $a_0 = 1, a_1 = 2$

$\Bigg($**Answer:** $a_n = (\sqrt{2})^n\left(\cos\frac{n\pi}{4} + \sin\frac{n\pi}{4}\right)\Bigg)$

Exercise 12-12: Solve the recurrence relation $a_{n+2} = 3a_{n+1} - 3a_n$, where $a_0 = 2, a_1 = 3$

$\Bigg($**Answer:** $a_n = 2(\sqrt{3})^n\cos\frac{n\pi}{6}\Bigg)$

Exercise 12-13: Solve the recursion relation

$$2a_{n+2} = -7a_{n+1} + 4a_n, \quad a_0 = 9, \ a_1 = -9$$

$\Bigg($**Answer:** $a_n = 3(-4)^n + 6\left(\frac{1}{2}\right)^n\Bigg)$

Exercise 12-14: Solve the recurrence relation $a_{n+2}-2a_{n+1}+a_n = 18, \qquad a_0 = 7,\ a_1 = 8$ for $n \geqslant 0$.

Solution: $x^2 - 2x + 1 = 0 \quad \Rightarrow \quad x = 1$ (Double Root)

$$a_n^h = c_1\,1^n + c_2\,n1^n = c_1 + c_2 n$$

$$a_n^p = A\,n^2\,1^n = A\,n^2$$

$$A(n+2)^2 - 2A(n+1)^2 + An^2 = 18 \quad \Rightarrow \quad A = 9$$

$$a_n = c_1 + c_2 n + 9n^2$$

$$n = 0 \quad \Rightarrow \quad c_1 = 7,$$

$$n = 1 \quad \Rightarrow \quad c_2 = -8$$

$$a_n = 7 - 8n + 9n^2$$

Exercise 12-15: Solve the following recursion relations:

a) $a_{n+2} = 6a_{n+1} - 9a_n, \quad a_1 = 15,\ a_2 = 126$

b) $a_{n+2} + a_{n+1} + a_n = 0, \quad a_0 = 4,\ a_1 = -5$

Solution:

a) $a_n = kr^n \quad \Rightarrow \quad r^2 - 6r + 9 = 0 \quad \Rightarrow \quad r = 3$

$a_n = k_1 3^n + k_2 n 3^n \quad \Rightarrow \quad 3k_1 + 3k_2 = 15,\ 9k_1 + 18k_2 = 126 \quad \Rightarrow \quad k_1 = -4,\ k_2 = 9$

$$\begin{aligned} a_n &= (-4)3^n + 9n3^n \\ &= 3^n(9n-4) \end{aligned}$$

b) $a_n = kr^n \quad \Rightarrow \quad r^2 + r + 1 = 0 \quad \Rightarrow \quad r = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}\,i$

$$a_n = k_1\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\,i\right)^n + k_2\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\,i\right)^n$$

$k_1 + k_2 = 4,\ k_1 - k_2 = 2\sqrt{3}i \quad \Rightarrow \quad k_1 = 2 + \sqrt{3}i,\ k_2 = 2 - \sqrt{3}i$

$$\begin{aligned} a_n &= (2+\sqrt{3}i)\,e^{\frac{2\pi n i}{3}} + (2-\sqrt{3}i)\,e^{\frac{-2\pi n i}{3}} \\ &= 4\cos\left(\frac{2\pi n}{3}\right) - 2\sqrt{3}\sin\left(\frac{2\pi n}{3}\right) \end{aligned}$$

Exercise 12-16: Solve the recursion relation $a_n = 10a_{n-1} - 18a_{n-2}$ with the initial conditions
$a_0 = 4, \quad a_1 = 20.$

Solution: $x^2 - 10x + 18 = 0$

$x = 5 \pm \sqrt{7}$

$a_n = c_1(5+\sqrt{7})^n + c_2(5-\sqrt{7})^n$

$$\begin{aligned} n &= 0 \quad \Rightarrow \quad & 4 &= c_1 + c_2 \\ n &= 1 \quad \Rightarrow \quad & 20 &= c_1(5+\sqrt{7}) + c_2(5-\sqrt{7}) \end{aligned}$$

$\Rightarrow \quad c_1 = c_2 = 2$

$a_n = 2\Big[(5+\sqrt{7})^n + (5-\sqrt{7})^n\Big]$

Exercise 12-17: Solve the recursion relation $2a_{n+2} - 8a_{n+1} + 7a_n = 0$ with the initial conditions
$a_0 = 16, \quad a_1 = 32.$

Solution: $2x^2 - 8x + 7 = 0$

$x = 2 \pm \frac{\sqrt{2}}{2}$

$a_n = c_1\left(2 + \frac{\sqrt{2}}{2}\right)^n + c_2\left(2 - \frac{\sqrt{2}}{2}\right)^n$

$$\begin{aligned} n &= 0 \quad \Rightarrow \quad & 16 &= c_1 + c_2 \\ n &= 1 \quad \Rightarrow \quad & 32 &= c_1\left(2 + \frac{\sqrt{2}}{2}\right) + c_2\left(2 - \frac{\sqrt{2}}{2}\right) \end{aligned}$$

$\Rightarrow \quad c_1 = c_2 = 8$

$a_n = 8\left[\left(2 + \frac{\sqrt{2}}{2}\right)^n + \left(2 - \frac{\sqrt{2}}{2}\right)^n\right]$

Week 13– Recurrence Relations and Algorithms

Exercise 13-1: Let $\Sigma = \{0, 1, 2\}$. Let a_n denote the number of n-bit strings that contain an even number of zeros. Clearly, $a_0 = 1, a_1 = 2, a_2 = 5$. What is a_n in general?

We can obtain an n-bit sequence with an even number of zeros as follows: Take an $(n-1)$–bit sequence with an even number of zeros, add 1 or 2. Or, take an $(n-1)$–bit sequence with an odd number of zeros and add 0. This means:

$$a_n = 2a_{n-1} + (3^{n-1} - a_{n-1}) \quad \Rightarrow a_n = \frac{3^n + 1}{2}$$

Exercise 13-2: Let $\Sigma = \{a, b\}$. How many words of length n are there that do not contain aa?

Let's call this number s_n. Clearly, $s_0 = 1, s_1 = 2, s_2 = 3$. The last two letters of our word must be ab or bb or ba. To obtain an n letter word that does not contain aa, we can add b to an $n-1$ letter word, or we can add ba to an $n-2$ letter word. So

$$s_n = s_{n-1} + s_{n-2}$$

$$s_n = \frac{5+3\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{5+3\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Exercise 13-3: In how many different ways can we express n as a sum of 1's and 2's, where order is important?

$$\left(\textbf{Answer: } a_n = a_{n-1} + a_{n-2}, \quad a_n = \frac{5+\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{5-\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

Exercise 13-4: Count the number of arithmetic expressions of n symbols made of $0, 1, \ldots, 9, +, *, /$.

$$\left(\textbf{Answer: } a_n = 10a_{n-1} + 29a_{n-2}, \quad a_n = \frac{5}{3\sqrt{6}}\left[(5+3\sqrt{6})^n - (5-3\sqrt{6})^n\right]\right)$$

Exercise 13-5: For $n \geqslant 1$ let a_n be the number of ternary strings (containing 0,1,2) of length n that do not contain 11 or 22. Find and solve a recurrence relation for a_n. (Hint: Consider those ending in 00 separately)

$$\left(\textbf{Answer: } a_n = 2a_{n-1} + a_{n-2}, \quad a_n = \frac{1}{2}\left[(1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}\right]\right)$$

Exercise 13-6: There are n houses around a circular street. We will color them using 5 distinct colors such that any two adjacent houses will have different colors. In how many different ways can we do this? (The recurrence relation is correct for $n \geqslant 2$)

$$\left(\textbf{Answer: } a_{n+2} = 3a_{n+1} + 4a_n, \quad a_n = 4^n + 4(-1)^n\right)$$

Exercise 13-7: Let $A = \{0, 01, 011, 111\}$. For $n \geqslant 1$, find the number of binary strings of length n made from elements of A.

$$\left(\textbf{Answer: } a_n = a_{n-1} + a_{n-2} + 2a_{n-2}, \quad a_n = \frac{4}{7}2^n + \frac{3}{7}\cos\frac{2n\pi}{3} + \frac{\sqrt{3}}{21}\sin\frac{2n\pi}{3}\right)$$

Exercise 13-8: Let a_n be the number of regions in the plane obtained by drawing n lines. For example, $a_1 = 2$, $a_2 = 4$, $a_3 = 7$.
(Lines are not parallel, three lines do not intersect at a point)

a) Find a recurrence relation for a_n.

b) Find a_n.

Solution:

a) $a_n = a_{n-1} + n$

b) $x - 1 = 0 \quad \Rightarrow \quad a_n^h = c$

$$a_n^p = An^2 + Bn$$

$$An^2 + Bn - A(n-1)^2 - B(n-1) = n$$

$$\Rightarrow \quad A = \frac{1}{2}, \quad B = \frac{1}{2}$$

$$a_n^p = \frac{n^2+n}{2} \quad \Rightarrow \quad a_n = \frac{n^2+n}{2} + c$$

$$a_1 = 2 \quad \Rightarrow \quad c = 1$$

$$a_n = \frac{n^2+n+2}{2}$$

Exercise 13-9: In how many different ways can we express 100 as a sum of 5's and 10's if order is important?

Solution: $a_n = a_{n-5} + a_{n-10}$

$x^{10} - x^5 - 1 = 0$

$x^5 = \dfrac{1 \pm \sqrt{5}}{2}$

$$a_5 = 1, \quad a_{10} = 2 \quad \Rightarrow \quad a_n = \frac{5+\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^{n/5} + \frac{5-\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)^{n/5}$$

$a_{100} = 10946$

Or, alternatively, divide by 5 and transform the problem to obtaining 20 using 1's and 2's.

Exercise 13-10: The number of laptops manufactured in a factory was 10000 in the first year, 16000 in the second year, and the average of previous two years after that. How many laptops were produced on the n^{th} year?

Solution: $a_n = \dfrac{a_{n-1} + a_{n-2}}{2}$

$2x^2 - x - 1 = 0$

$x = 1$ or $x = -\dfrac{1}{2}$

$a_n = c_1(1)^n + c_2\left(-\dfrac{1}{2}\right)^n$

$10000 = c_1 - \dfrac{c_2}{2}$

$16000 = c_1 + \dfrac{c_2}{4}$

$a_n = 14000 + 8000\left(-\dfrac{1}{2}\right)^n$

Exercise 13-11: An algorithm makes $T(n)$ operations to solve a problem of input size n. Find an explicit formula for $T(n)$ if

a) $T(n) = T(n/2)$

b) $T(n) = 2T(n/2)$

c) $T(n) = aT(n/b)$
(Hint consider the transformation $n = 2^m$)

Week 14– Nonhomogeneous Recurrence Relations

Consider the second order nonhomogeneous relation

$$a_n + pa_{n-1} + qa_{n-2} = kr^n$$

- Solve the homogeneous version of the equation $a_n + pa_{n-1} + qa_{n-2} = 0$. Call the solution a_n^h.
- Let a_n^p be any particular solution for the nonhomogeneous equation.
    1. If r is not a root of the homogeneous equation, $a_n^p = Cr^n$.
    2. If r is a single root of the homogeneous equation, $a_n^p = Cnr^n$.
    3. If r is a double root of the homogeneous equation, $a_n^p = Cn^2r^n$.
- The general solution is: $a_n = a_n^h + a_n^p$.

Exercise 14-1: Solve the recurrence relation $a_{n+2} - a_{n+1} - 6a_n = 14 \cdot 5^n, \quad a_0 = 5,\ a_1 = 10$ for $n \geqslant 0$.

$\left(\textbf{Answer: } a_n = \frac{7}{5}(-2)^n + \frac{13}{5}3^n + 5^n\right)$

Exercise 14-2: Solve the recurrence relation $a_{n+2} - 5a_{n+1} + 4a_n = 6 \cdot 4^n, \quad a_0 = 2,\ a_1 = 6$ for $n \geqslant 0$.

$\left(\textbf{Answer: } a_n = \frac{2}{3}4^n + \frac{4}{3} + \frac{1}{2}n4^n\right)$

Exercise 14-3: Solve the recurrence relation $a_{n+2} - 16a_{n+1} + 64a_n = 2 \cdot 8^{n+2}$, $a_0 = 1,\ a_1 = 12$ for $n \geqslant 0$.

$\left(\textbf{Answer: } a_n = \left(1 - \frac{1}{2}n + n^2\right)8^n\right)$

Exercise 14-4: Solve $a_0 = 1,\ a_1 = 2$ and $a_n = 2a_{n-1} - a_{n-2} + 2$ for $n \geqslant 2$.

$\left(\textbf{Answer: } a_n = n^2 + 1\right)$

Exercise 14-5: Solve the recurrence relation $a_{n+2} - 4a_{n+1} + 3a_n = -200, \quad a_0 = 3000,\ a_1 = 3300$ for $n \geqslant 0$.

$\left(\textbf{Answer: } a_n = 100 \cdot 3^n + 2900 + 100n\right)$

Generalization of the Method

We can solve recursion relations with the right hand side given below using the same method:

$f(n)$	a_n^p candidate
1	a
n	$an + b$
n^2	$an^2 + bn + c$
n^k	$a_k n^k + \cdots + a_0$
r^n	ar^n
$n^k r^n$	$r^n(a_k n^k + \cdots + a_0)$

If we have a characteristic root at a_n^p, we have to multiply it by n for a single root, and n^2 for a double root.

Exercise 14-6: Solve the recurrence relation $2a_{n+2} - 9a_{n+1} + 9a_n = 10 \cdot 4^n, \qquad a_0 = 3,\ a_1 = 14$
for $n \geqslant 0$.

Solution:

$$2x^2 - 9x + 9 = 0 \quad \Rightarrow \quad x = 3,\ \frac{3}{2}$$

$$a_n^h = c_1\, 3^n + c_2 \left(\frac{3}{2}\right)^n \qquad a_n^p = A\, 4^n$$

$$2A4^{n+2} - 9A4^{n+1} + 9A4^n = 10 \cdot 4^n \quad \Rightarrow \quad A = 2$$

$$c_1 + c_2 + 2 = 3, \quad 3c_1 + \frac{3}{2}c_2 + 8 = 14 \quad \Rightarrow \quad c_1 = 3, c_2 = -2$$

$$a_n = 3 \cdot 3^n - 2\left(\frac{3}{2}\right)^n + 2 \cdot 4^n$$

Exercise 14-7: Solve the recurrence relation $a_{n+2} - 4a_n = \frac{16}{3}\, 2^n, \qquad a_0 = 3,\ a_1 = -2$
for $n \geqslant 0$.

Solution:

$$x^2 - 4 = 0 \quad \Rightarrow \quad x = 2,\ -2$$

$$a_n^h = c_1\, 2^n + c_2\, (-2)^n \qquad a_n^p = A\, n\, 2^n$$

$$A(n+2)2^{n+2} - 4An2^n = \frac{16}{3} \cdot 2^n \quad \Rightarrow \quad A = \frac{2}{3}$$

$$c_1 + c_2 = 3, \quad 2c_1 - 2c_2 + \frac{4}{3} = -2 \quad \Rightarrow \quad c_1 = \frac{2}{3}, c_2 = \frac{7}{3}$$

$$a_n = \frac{7}{3}\,(-2)^n + \frac{2}{3}(n+1)\, 2^n$$

Exercise 14-8: Solve the recurrence relation $a_{n+2} - 3a_{n+1} + 2a_n = -5, \qquad a_0 = 4,\ a_1 = 12$
for $n \geqslant 0$.

Solution:

$$x^2 - 3x + 2 = 0 \quad \Rightarrow \quad x = 2,\ 1$$

$$a_n^h = c_1 + c_2 2^n \qquad a_n^p = A\, n$$

$$A(n+2) - 3A(n+1) + 2An = -5 \quad \Rightarrow \quad A = 5$$

$$c_1 + c_2 = 4, \quad c_1 + 2c_2 = 7 \quad \Rightarrow \quad c_1 = 1, c_2 = 3$$

$$a_n = 1 + 5n + 3 \cdot 2^n$$

Exercise 14-9: Solve the recurrence relation $a_{n+2} - 14a_{n+1} + 49a_n = -6 \cdot 7^{n+1}, \qquad a_0 = 1,\ a_1 = 11$
for $n \geqslant 0$.

Solution:

$$x^2 - 14x + 49 = 0 \quad \Rightarrow \quad x = 7 \text{ (Double Root)}$$

$$a_n^h = c_1\, 7^n + c_2\, n7^n \qquad a_n^p = A\, n^2\, 7^n$$

$$A(n+2)^2 7^{n+2} - 14A(n+1)^2 7^{n+1} + 49An^2 7^n = -42 \cdot 7^n \quad \Rightarrow \quad A = -\frac{3}{7}$$

$$c_1 = 1, \quad -3 + 7 + 7c_2 = 11 \quad \Rightarrow \quad c_2 = 1$$

$$a_n = \left(1 + n - \frac{3}{7}n^2\right) 7^n$$

Exercise 14-10: Solve the recursion relation

$$3a_{n+2} = -17a_{n+1} + 6a_n, \quad a_0 = 13,\ a_1 = -21$$

$\left(\textbf{Answer: } a_n = 4(-6)^n + 9\left(\frac{1}{3}\right)^n\right)$