Çankaya University
Department of Computer Engineering
2014 - 2015 Fall Semester

CENG 277 - Discrete Structures

Final Examination

1) Show that if $x > -1$ and n is a positive integer,

$$(1+x)^n \geqslant 1 + nx$$

2) There are 101 chairs arranged in a row. n people sit on these chairs. We are certain that, however they are arranged, there are always 5 or more empty consecutive chairs. What is n?

3) Consider set of 5 letter strings made of the letters in the set $\{a, b, c, d, e, f, g\}$. Find the number of strings where at least one letter is repeated exactly twice. (For example, *egbcb* or *cacee*)

4) Find the asymptotic complexity of the MAIN program, given that the complexity of HELPER program is $\Theta(n)$. (You may assume n is a power of 2)

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—MAIN—
INPUT: integer n
i = 1
While i ⩽ n
    For j = 1 to i
        HELPER(j)
    EndFor
    i = 2 ∗ i
EndWhile
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5) Let a_n denote the number of ways we can express the integer n as a sum of odd integers where order is important. For example, $a_5 = 5$ and $a_6 = 8$ because 5 can be written as:

$$1+1+1+1+1 = 3+1+1 = 1+3+1 = 1+1+3 = 5$$

and 6 can be written as:

$$1{+}1{+}1{+}1{+}1{+}1 = 3{+}1{+}1{+}1 = 1{+}3{+}1{+}1 = 1{+}1{+}3{+}1 = 1{+}1{+}1{+}3 = 5{+}1 = 1{+}5 = 3{+}3$$

a) Find a recursion relation for a_n.

b) Solve the recurrence relation.

Answers

1) Let $n = 1$. Clearly, $1 + x = 1 + x$, it is correct for base case.

Now assume the inequality is correct for $n = k$.

$$(1 + x)^k \geqslant 1 + kx$$

$$(1 + x) = (1 + x)$$

Multiply them side by side to obtain

$$(1 + x)^{k+1} \geqslant 1 + (k + 1)x + kx^2$$

kx^2 is positive therefore

$$(1 + x)^{k+1} \geqslant 1 + (k + 1)x$$

So the claim is correct by Mathematical Induction.

2) Arrange the chairs in groups of 5. There will be 20 groups and one extra. We want 5 empty slots in at least one group, so by pigeonhole principle, we need at least $4 \cdot 20 + 1 + 1 = 82$ empty slots. This means at most $101 - 82 = 19$ people.

3) We will first choose letters, then arrange them. The possibilities are:

$$2+1+1+1 \quad : \quad \binom{7}{1}\binom{6}{3}\frac{5!}{2!} \quad = \quad 8400$$

$$2+2+1 \quad : \quad \binom{7}{2}\binom{5}{1}\frac{5!}{2!\,2!} \quad = \quad 3150$$

$$2+3 \quad : \quad \binom{7}{1}\binom{6}{1}\frac{5!}{2!\,3!} \quad = \quad 420$$

$$8400 + 3150 + 420 = 11970$$

Second Method

Choose one letter, choose two slots, put it in two slots. Fill other slots randomly using the other 6 letters. This guarantees we have exactly two of at least one letter, but if we have two of two distinct letters, we are counting the word twice. So subtract them.

$$7 \cdot \binom{5}{2} \cdot 6^3 - 3150 = 11970$$

4) The number of operations is:

$$1 + (1+2) + (1+2+3+4) + (1+2+\cdots+8) + \cdots + (1+2+\cdots+n)$$

$$\frac{1\cdot 2}{2} + \frac{2\cdot 3}{2} + \frac{4\cdot 5}{2} + \cdots + \frac{n(n+1)}{2}$$

Assuming n is a power of 2, we obtain

$$\frac{2n^2+3n-2}{3}$$

operations, so it is $\Theta(n^2)$.

Second Method

An easier way to see this is as follows: The For loop is $\Theta(i^2)$. Its maximum run is of n^2 operations, then it decreases by a factor of 4. So we have to add:

$$n^2 + \frac{n^2}{4} + \frac{n^2}{16} + \cdots$$

$$= \frac{4}{3}n^2 = \Theta(n^2)$$

5) Consider all valid orders for a_n. Either they end with $+1$ or they end with $+k$ where $k \geqslant 3$. If it ends with $+1$, remove that and obtain a_{n-1}. If it ends with $+k$, replace it by $k-2$ and obtain a_{n-2}. As there is no other possibility, we obtained:

$$a_n = a_{n-1} + a_{n-2}$$

Using the initial conditions $a_1 = 1$ and $a_2 = 1$, we obtain the Fibonacci Sequence,

$$a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$