Çankaya University
Department of Computer Engineering
2014 - 2015 Fall Semester

CENG 277 - Discrete Structures
Second Midterm Examination

1) **a)** Define a relation R in the set of positive integers as follows: $(n, m) \in R$ if n and m are relatively prime. ($\gcd(n, m) = 1$) Is this an equivalence relation?

b) How many onto functions are there from a 6 element set to a 4 element set?

2) Let $S = \{1, 2, 3, \ldots, 21\}$. Show that if we choose any 12 numbers from the set S, there will be at least 2 numbers among the 12 such that their sum is 23.

3) In how many different ways can we distribute 33 identical balls to 5 distinct kids such that each one gets at most 10 balls?

4) We will buy 50 bottles of fruit juice. Available types are: Apricot, Peach, Orange, Cherry and Grapefruit. We want

- At most 10 Apricots,
- 5 or 6 Peaches,
- 10 or 11 Oranges,
- Even number of Cherries.
- Odd number of Grapefruits.

In how many different ways can we do this?

5) There are 40 houses on a street. We will paint them with one of 5 colors: Blue, Green, Red, White, Yellow. We want at least one blue and at least two green houses.

In how many different ways can we do this?

Answers

1) a) No, because it is not transitive.

For example $\gcd(4,9)=1$, $\gcd(9,22)=1$ but $\gcd(4,22)\neq 1$

b) We have to use Stirling numbers. The result is $4!S(6,4)=24\cdot 65=1560$
Or, we can use inclusion-exclusion. Then we obtain: $4^6-4\cdot 3^6+6\cdot 2^6-4\cdot 1^6=1560$

2) Put the numbers except 1 in groups of two as follows: $\{2,21\},\{3,20\},\ldots,\{11,12\}$
There are 10 groups. Now choose 12 numbers. One of them could be 1. In that case, we are choosing 11 numbers from 10 groups. (Otherwise, we are choosing 12 numbers from 10 groups) By pigeonhole principle, we have to choose two numbers from at least one group.

3) Distribute randomly in $\binom{33+5-1}{33}=\binom{37}{33}$ different ways. Then subtract cases where one kid gets 11 or more: $\binom{5}{1}\binom{22+5-1}{22}=5\binom{26}{22}$ Similarly, add cases where 2 different kids get 11 or more and subtract cases where 3 kids get 11.

$$\binom{37}{4}-5\binom{26}{4}+\binom{5}{2}\binom{15}{4}-\binom{5}{3}\binom{4}{4}=4935$$

4) Using Generating Functions, we obtain

$$(1+x+\cdots+x^{10})(x^5+x^6)(x^{10}+x^{11})(1+x^2+x^4\cdots)(x+x^3+x^5\cdots)$$

$$=\frac{1-x^{11}}{1-x}\,x^5(1+x)\,x^{10}(1+x)\,\frac{1}{1-x^2}\,\frac{x}{1-x^2}$$

$$=\frac{x^{16}-x^{27}}{(1-x)^3}$$

$$=(x^{16}-x^{27})\sum_{n=0}^{\infty}\binom{n+2}{n}x^n$$

The coefficient of x^{50} is:

$$\binom{36}{34}-\binom{25}{23}=330$$

5) Using Exponential Generating Functions, we obtain:

$$(e^x-1)(e^x-1-x)e^{3x}=e^{5x}-2e^{4x}+e^{3x}+x(e^{3x}-e^{4x})$$

The coefficient of $\frac{x^{40}}{40!}$ is:

$$5^{40}-2\cdot 4^{40}+3^{40}+40\cdot 3^{39}-40\cdot 4^{39}$$